STABILITY OF A BREATHING K-V BEAM *

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Abstract

Many explanations of halo formation in high current ion beams require the existence of particles which are outside the beams central core. We propose a mechanism which is capable of moving some particles outside the core. Specifically, we consider a 2-D KV beam which is started into a uniform density breathing oscillation by some mismatch in the transverse focusing pattern. We then consider perturbations with non-linear space charge density and find that they can be unstable against small oscillations for certain ranges of mismatch and tune depression. The stability limits in the mismatch/tune depression space have been computed for the first three azimuthally symmetric modes with non-linear charge density. It appears that even modest values of mismatch and tune depression can lead to instabilities which are capable of moving particles outside the core of the beam.

I. INTRODUCTION

Interest has arisen recently in using ion linacs in high current applications. In order to keep the beam loss to the order of 1 ppm to avoid serious linac activation, it is necessary to understand emittance growth and halo formation in great detail in order to produce an acceptable design.

Accordingly, recent attention has been focused on understanding the mechanism(s) by which halos are produced. This includes a review of observations and related simulations[1], a variety of simulations and experiments[2], [3]. Several models have been constructed to explore resonances between particle oscillation frequencies and the periodicity of the focusing system or core oscillation modes[4], [5]. Many of the simulations show the onset of chaotic motion at high space charge levels.

In a recent publication[6], we proposed a simple model in which a K-V beam, excited into a uniform density “breathing” mode by some sort of mismatch, interacts resonantly with individual oscillating ions. If the ions find themselves outside the core for part of their oscillation, the resulting non-linearity of the ion oscillations can lead to a phase lock with the breathing oscillation, producing a halo whose parameters can be predicted and whose appearance matches that in simulations performed by Wangler and by Ryne [7]. The unanswered question is: “What is the mechanism by which ions initially escape from the core in order to participate in the formation of the halo?”

Obviously, any unstable longitudinal or transverse collective modes involving the core are capable of moving particles outside the core. Studies of the transverse stability of a matched K-V beam[8], [9] have shown that instabilities exist for tune depressions (ratio of ion oscillation frequency with space charge to that without space charge) of 0.4 or less. In the present paper, we analyze the instabilities of a breathing K-V beam for various collective modes involving non-uniform charge density and find that modes involving a significant breathing amplitude will be unstable at tune depressions as high as 0.7 or 0.8.

*Work Supported by U.S. Department Of Energy.

II. BREATHING MODE

The envelope equation of a KV beam is

$$a'' + k^2 a = \frac{I}{a} + \frac{\epsilon^2}{a^3}, \tag{1}$$

where a is the beam radius, the prime stands for d/dz, k is the tune due to the external linear restoring force, I is the perveance defined by $I = eI_0 Z_0 c/2\pi m v_0^3$, and where $\pi\epsilon$ is the transverse emittance of the beam. Here $Z_0 = 120\pi$ ohms is the impedance of free space, I_0 is the beam current and v_0 is the particle's axial velocity. We assume that k^2 is independent of z in the present work. If we start with $a(0) = a_1$, $a'(0) = 0$, an integral of Eq. (1) gives

$$a'^2 = 2I\ell n\frac{a}{a_1} + k^2(a_1^2 - a^2) + \epsilon^2\left(\frac{1}{a_1^2} - \frac{1}{a^2}\right), \tag{2}$$

which enables us to obtain the other value of $a(\equiv a_2)$ at which $a' = 0$, as well as the period of the breathing motion.

We now set $a^2 = \beta\epsilon$, and change the independent and dependent variables from z, x, y to $\phi = \int dz/\beta$, $u(\phi) = x(z)/\sqrt{\beta\epsilon}$, $v(\phi) = y(z)/\sqrt{\beta\epsilon}$, such that

$$\ddot{u} + u = 0\ \ ,\ \ \ddot{v} + v = 0, \tag{3}$$

where the dot denotes derivative with respect to ϕ. Thus the breathing mode can be described by specifying β as a function of ϕ, with period ϕ_0. The transformation clearly depends on the size of the “mismatch”, that is, on the relative amplitude of the breathing oscillation. If we scale β so that $\beta(\phi) = \sigma(\phi)/k$ and define $\alpha = I/k\epsilon$, The envelope equation can be written as

$$\frac{\ddot{\sigma}}{2\sigma} = 1 + \alpha\sigma - \sigma^2 + \frac{3}{4}\frac{\dot{\sigma}^2}{\sigma^2}. \tag{4}$$

We note that a matched beam (zero breathing amplitude) has the matched amplitude $\sigma_0 = \alpha/2 + \sqrt{1 + \alpha^2/4}$ and that the tune depression for a matched beam is given by $\eta \equiv \sqrt{k^2 - I/a_0^2}/k = 1/\sigma_0 = \sqrt{1 + \alpha^2/4} - \alpha/2$.

III. PHASE SPACE DISTRIBUTION

We now wish to consider small perturbations from a uniform charge density breathing mode in the phase space distribution. For this purpose, we use the variables $u(\phi)$, $v(\phi)$ and ϕ and write

$$f(u, v, \dot{u}, \dot{v}, \phi) = f_0(u, v, \dot{u}, \dot{v}) + f_1(u, v, \dot{u}, \dot{v}, \phi), \tag{5}$$

where the unperturbed distribution (including the breathing mode) is

$$f_0(u, v, \dot{u}, \dot{v}) = (\tau_0/\pi^2)\delta(u^2 + v^2 + \dot{u}^2 + \dot{v}^2 - 1). \tag{6}$$

Here $\tau_0 = I_0/v_0$ is the line charge density of the beam. The charge density (in the x, y space) is then

$$\rho_0 = \frac{I_0}{\pi v_0 \beta\epsilon} \left\{ \begin{array}{ccc} 1 & , & u^2+v^2<1 \\ 0 & , & u^2+v^2>1 \end{array} \right\} \qquad (7)$$

and

$$\rho_1 = \frac{1}{\beta\epsilon} \int d\dot{u} \int d\dot{v}\; f_1(u,v,\dot{u},\dot{v},\phi). \qquad (8)$$

We assume that the electric field due to ρ_1 is derivable from a scalar potential $G(u,v,\phi)$ such that

$$\nabla\cdot\vec{E}_\perp = \frac{1}{\sqrt{\beta\epsilon}}\left(\frac{\partial E_x}{\partial u}+\frac{\partial E_y}{\partial v}\right) = \frac{\rho_1}{\epsilon_0}$$
$$= -\frac{1}{\beta\epsilon}\left(\frac{\partial^2 G}{\partial u^2}+\frac{\partial^2 G}{\partial v^2}\right) = \frac{1}{\beta\epsilon\epsilon_0}\int d\dot{u}\int d\dot{v} f_1. \qquad (9)$$

The equations of motion, including the force due to the non-uniform charge distribution, are

$$\ddot{u}+u = -\frac{e}{mv_0^2}\frac{\beta}{\epsilon}\frac{\partial G}{\partial u} \;\; , \;\; \ddot{v}+v = -\frac{e}{mv_0^2}\frac{\beta}{\epsilon}\frac{\partial G}{\partial v}. \qquad (10)$$

If we now write

$$f_1(u,v,\dot{u},\dot{v},\phi) = g(u,v,\dot{u},\dot{v},\phi) f_0'(u^2+v^2+\dot{u}^2+\dot{v}^2), \quad (11)$$

keeping only terms linear in f_1 or ρ_1 (or G), the Vlasov equation becomes

$$\frac{\partial g}{\partial\phi}+\dot{u}\frac{\partial g}{\partial u}+\dot{v}\frac{\partial g}{\partial v}-u\frac{\partial g}{\partial\dot{u}}-v\frac{\partial g}{\partial\dot{v}} = R(u,v,\dot{u},\dot{v},\phi), \quad (12)$$

where the right hand side is

$$R(u,v,\dot{u},\dot{v},\phi) = \frac{2e}{mv_0^2}\frac{\beta}{\epsilon}\left[\dot{u}\frac{\partial G}{\partial u}+\dot{v}\frac{\partial G}{\partial v}\right]. \qquad (13)$$

Equations (9) and (12) are coupled integro-differential equations. Since the operator on the left side of Eq. (12) corresponds to the sinusoidal orbits in Eq. (3), Eq. (12) has a formal solution which can be written as

$$g(u,v,\dot{u},\dot{v},\phi) = \int_{-\infty}^{\phi} d\psi R(u',v',\dot{u}',\dot{v}',\psi), \qquad (14)$$

where $u' = uc-\dot{u}s$, $v' = vc-\dot{v}s$, $\dot{u}' = \dot{u}c+us$, $\dot{v}' = \dot{v}c+vs$, with $c\equiv\cos(\phi-\psi)$, $s\equiv\sin(\phi-\psi)$.

We now proceed, as in the analysis for a matched K-V beam[8], to guess at the form of the potential $G(u,v,\phi)$ and to determine the perturbed phase space distribution $g(u,v,\dot{u},\dot{v},\phi)$ using Eq. (14). Using Eqs. (11) and (9), we then obtain $\partial^2 G/\partial u^2 + \partial^2 G/\partial v^2$ and require that it agree with our guess for G.

Remarkably, our guess, which is almost identical to the form used for the matched K-V beam, works once again.

We now conjecture that $G(u,v,\phi)$ is

$$G(u,v,\phi) = P(\phi)F(u,v) \qquad (15)$$

with

$$F(u,v) = (u+iv)^m\,{}_2F_1(-j,\; m+j;\; m+1;\; u^2+v^2)$$
$$= d_{jm}\sum_\ell \frac{(-1)^\ell(m+j+\ell-1)!}{\ell!(m+\ell)!(j-\ell)!}(u+iv)^{\ell+m}(u-iv)^\ell, \quad (16)$$

where $d_{jm} = j!\; m!/(m+j-1)!$. Here $P(\phi)$ is a function periodic in ϕ (with period ϕ_0, the same as that of the breathing oscillation) which is yet to be determined. The corresponding charge density, according to Eq. (9), is

$$\rho_1(u,v,\phi) = \frac{4\epsilon_0}{\epsilon}\frac{P(\phi)}{\beta(\phi)}d_{jm}\sum_\ell\frac{(-1)^\ell(m+j+\ell)!}{\ell!(m+\ell)!(j-1-\ell)!}\times$$
$$(u+iv)^{\ell+m}(u-iv)^\ell, \qquad (17)$$

with m and $j-1$ being the number of azimuthal and radial nodes in the perturbed charge density.

Requiring the self-consistency of Eqs. (9) and (14), we obtain an integral equation for $P(\phi)$

$$P(\phi) = -\alpha\int_{-\infty}^{\phi} d\psi P(\psi)\sigma(\psi)\frac{\partial Q}{\partial\psi}, \qquad (18)$$

where

$$Q(\phi-\psi) = \sum_r\frac{(-1)^{j+r}(m+j+r-1)!}{r!(m+r)!(j-r)!}\cos^{m+2r}(\phi-\psi)\times$$
$$= (-1)^j[d_{jm}]^{-1}\cos^m(\phi-\psi)\times$$
$${}_2F_1(-j,m+j;m+1;\cos^2(\phi-\psi)). \qquad (19)$$

To recapitulate, we have confirmed that the conjecture for the electrostatic potential in Eq. (15) leads to a perturbed phase space density in Eq. (11) which reproduces the perturbed space charge density corresponding to the potential in Eq. (15), provided $P(\phi)$ satisfies the integral equation in Eq. (18).

IV. DIFFERENTIAL EQUATION FOR $P(\phi)$

The integral equation for $P(\phi)$ in Eq. (18) can be converted to a linear differential equation with periodic coefficients. As an illustration, we consider the case $j = 2$, $m = 0$, and take successive derivatives of Eq. (18) with respect to ϕ, obtaining contributions from both the upper limit of the integral and the integrand. And then we construct a linear combination of $P(\phi)$, $\ddot{P}(\phi)$ and $P^{iv}(\phi)$ in which the integrals cancel. Specifically

$$P^{iv}+(20+2\alpha\sigma)\ddot{P}+4\alpha\dot{\sigma}\dot{P}+(64-4\alpha\sigma+2\alpha\ddot{\sigma})P=0. \quad (20)$$

Since $\sigma(\phi)$ in Eq. (4) is a periodic function of ϕ with period ϕ_0, Eq. (20) is a Mathieu-like equation for $P(\phi)$. If we let V be the four-component vector $(P,\dot{P},\ddot{P},\dddot{P})$, Eq. (20) can then be written as the single 4×4 matrix equation $\dot{V} = TV$ where the matrix T depends on ϕ because σ depends on ϕ.

For general j and m, by taking $2j+m$ or $2j+m+1$ derivatives of $P(\phi)$, it is always possible to construct a linear combination which eliminates all the integrals, as we did in Eq. (20). The order of the resulting differential equation is $2j+m$ for m even or $2j+m+1$ for m odd, as is also the dimension of the vector V and the matrix T.

V. NUMERICAL STUDIES

To determine the stability of a specific mode of density perturbation, we first need to solve the equation of the envelope oscillation shown in Eq. (4) numerically. With the solutions of $\sigma(\phi)$, we can numerically integrate the matrix form of the differential equation for $P(\phi)$. The eigenvalues of the transfer matrix $\mathcal{T}$ for a breathing period then determines the stability of the mode denoted by j, m for the space charge α and the mismatch parameter σ_1. Specifically, the mode will be unstable if the absolute value of any of the eigenvalues of $\mathcal{T}$ is greater than 1.

Starting from the integral equation for $P(\phi)$ in Eq. (18), we can also make the transformation to differential equations for $(j, m) = (3, 0)$ and (4,0),and thus determine their stabilities with respect to different parameters.

As for the matched beam, i.e., $\sigma = \sigma_0 =$ constant, the stability limits of the modes $(j, m) = (2, 0), (3, 0)$ and (4,0), are where $\eta_{\rm limit}$ = 0.2425, 0.3859 and 0.3985 respectively. In fact, $m = 0$ is the most restrictive mode for all m, and $j = 4$ is the most serious mode that gives the the largest threshold value of η, i.e. the smallest space charge limit, for all $(j, 0)$ modes[8]. Therefore, the (4,0) mode is the least stable mode for the space charge limit of a KV beam. In Figs. 1, we show the stability diagram for these three cases in the $\mu - \eta$ space, where $\mu \equiv a_1/a_0$. The values of $\eta_{\rm limit}$ on the $\mu = 1$ axis for each case is confirmed in the figures.

The cusps appearing in these diagrams are caused by resonances of the mode frequency. In Fig. 1(a), the deep fissure down to the matched parameter $\mu = 1$ is where the phase advance of the (2,0) mode oscillation during one period of the breathing mode is π, when $\eta_{\rm limit} = 0.5235$. Note that for this resonance the breathing frequency is twice the mode frequency. We believe that the other slits appearing in the stable domains are also due to resonance for particular parameters of tune depression and mismatch. As for the higher modes (3,0) and (4,0), the $\phi = \pi$ resonance occurs outside of their stability limits. That is why the deep fissure that meets the $\mu = 1$ axis is not seen in either the $j = 3$ or the $j = 4$ cases. One can also see that, as j increases, not only does $\eta_{\rm limit}$ moves "backward", i.e., to smaller space charge, but also the stable band width for the mismatch parameter μ becomes narrower. This implies, at least up to $j = 4$ for a KV beam, that the area of stability decreases as j increases.

We are currently using numerical orbit simulations to confirm the stability regions shown in Fig. 1.

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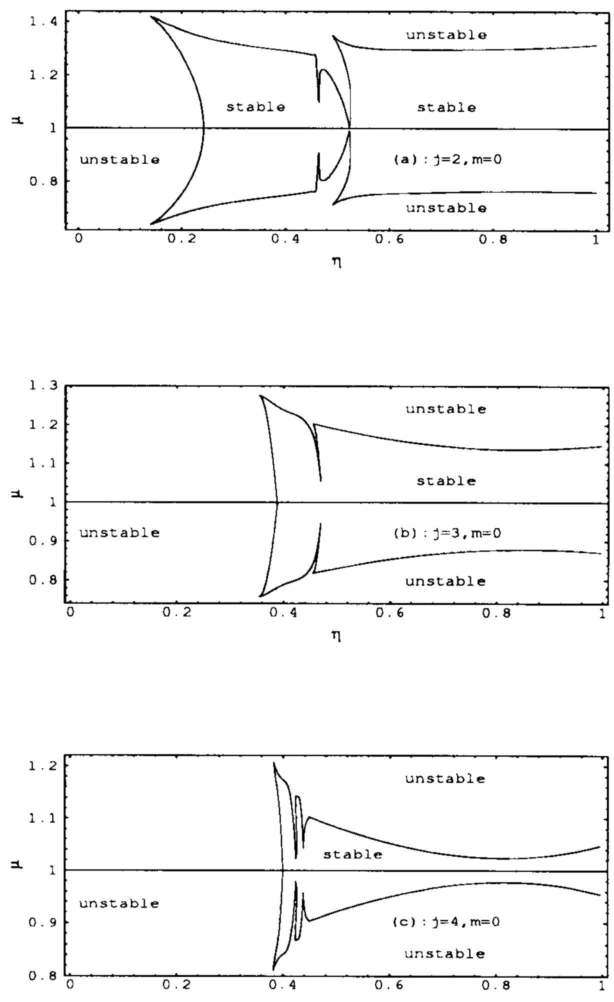

Figure. 1. Stability diagram of $\mu - \eta$ space, for (a)$j = 2, m = 0,$ (b)$j = 3, m = 0, and$ (c)$j = 4, m = 0.$