*RADIAL MODE EVOLUTION IN LONGITUDINAL BUNCHED BEAM INSTABILITY**

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I. INTRODUCTION

An indispensable aspect of the bunched beam instability mechanism is the variation of the particle distribution with respect to the beam intensity. This density variation can be shown as the evolution of radial modes. The radial modes, which are determined by the stationary particle distribution and the impedance, represent the coherence of the particle density variation governed by the Vlasov equation. Using this coherence in the beam instability analysis gives rise to not only the computational efficiency but also the physical insight into the instability mechanism. The evolution of the radial modes displays several interesting properties for the cases without and with synchrotron frequency spread. If the azimuthal mode coupling cannot be neglected, then corresponding to each coherent frequency shift there exists an extended radial mode which includes the interactions from other azimuthal modes. In this article, the radial mode evolution and the related physical implications will be discussed, which are useful for the understanding of the beam instabilities, and also useful for the beam diagnostics.

II. RADIAL MODE

If the azimuthal mode coupling is neglected, then the Sacherer integral equation can be written as an eigenvalue problem [1-4],

$$(\omega - m\omega_s)\alpha^{(m)} = M^{(m)}\alpha^{(m)} \qquad (1)$$

where ω is the coherent frequency shift, m is the azimuthal mode number, ω_s is the synthrotron frequency. The system matrix $M^{(m)}$ represents a feedback determined by the particle distribution of the beam and the impedance, and it is proportional to the beam intensity. The eigenvector $\alpha^{(m)} = \left[\alpha_0{}^{(m)} \cdots \alpha_{\bar{k}}{}^{(m)}\right]^T$, where $\bar{k}$ denotes truncation and the superscript T denotes transpose, represents the radial mode defined on the basis of the normalized orthogonal polynomials $f_k{}^{(m)}(r)$,

$$R^{(m)}(r) = W(r)\sum_{k'=0}^{\infty}\alpha_{k'}{}^{(m)} f_{k'}{}^{(m)}(r) \qquad (2)$$

Note that the weight function is defined as $W(r) = -d\psi_0/(dr \cdot r)$, where ψ_0 is the stationary particle distribution. The radial mode together with the rotation factor e^{jm0} determine the perturbation particle distribution in the phase space,

$$\psi_p(r,0) = \sum_{m'=-\infty}^{\infty} R^{(m')}(r)e^{jm'0} \qquad (3)$$

The projection of this perturbation on the phase deviation axis is the line density, which can be observed to obtain the information of the radial mode variation.

Given stationary distribution and impedance, the radial modes represent coherent particle density evolution, governed by the Vlasov equation. This coherence implies the discreteness of the modes and the corresponding frequency shifts. The orthogonal polynomials used in [2,3], i.e. for a Gaussian distribution, the generalized Laguerre polynomials, and for a parabolic, the Jacobi, can guarantee the convergence. In Fig. 1, the weighted orthogonal polynomials for aGaussian distribution and the Hankel spectra of these orthogonal polynomials are shown for the azimuthal mode m=1. We may observe that the spectra of the polynomials cover from the low frequency to the high, therefore, the larger dimension expansion of the radial modes implies to include higher frequency components. For some impedances such as the RF cavity, the convergence is very fast. Usually an expansion of a few dimension gives rise to a good approximation, and a few radial modes represent the whole system well.

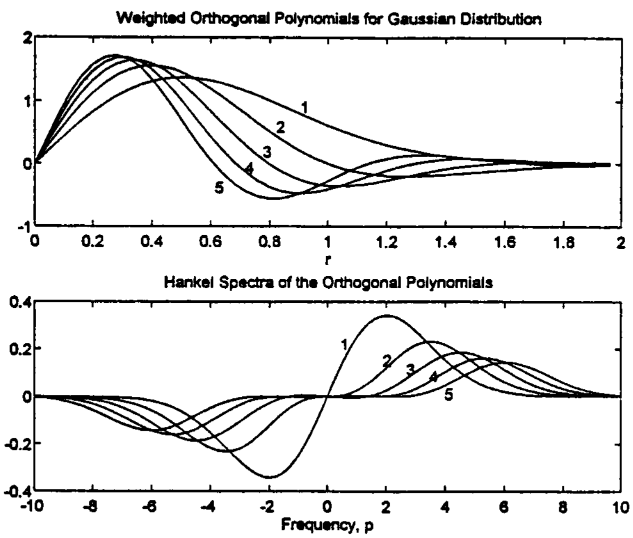

Fig. 1 Orthogonal Polynomials and Spectra

Using the coherence of the particle density evolution, solving of the eigenvalue problem in (1) is equivalent to decoupling the system into several independent subsystems, each one with its own inherent frequency and representation of its radial mode.

In general, different orthogonal basis can be chosen for the radial mode expansion, the convergence will however be affected. The problem will be discussed in Section V.

Several properties associated with the radial mode are mentioned at this point. 1) For a non-zero beam intensity, the coherent frequency shift ω is linearly proportional to the intensity. 2) The eigenvectors and therefore the radial modes are invariant with respect to the intensity. 3) The amplitude of the radial mode is arbitrary, since the scaling of the eigenvector is arbitrary. 4) The damping or antidamping of these modes are independent for each other.

The eigenvalue problem shown in (1) is the simplest one in the perturbation problem, it also has a most popular application, especially in a regime of low intensity. It, however, cannot satisfy the situations with synchrotron frequency spread, or if the radial modes are affected by other azimuthal modes, i.e. mode coupling, or if due to high intensity the potential well is distorted. Under these conditions, equation (1) needs to be modified.

III. FREQUENCY SPREAD

For the synchrotron frequency spread due to the RF nonlinearity, the equation (1) is modified as,

$$\left(\omega = m\omega_{SC}\right)\alpha^{(m)} = \left(mN^{(m)} + M^{(m)}\right)\alpha^{(m)} \tag{4}$$

where $N^{(m)}$ represents the frequency dispersion, which does not depend on the beam intensity. This matrix introduces no instability mechanism, and it is dominant at low intensity. At high intensity the dominance is transferred to the matrix $M^{(m)}$, and the transition process shows the Landau damping.

We note that some properties associated with the radial modes are changed. 1) The eigenvalues are no longer linearly proportional to the beam intensity. 2) The radial modes are no longer invariant to the intensity.

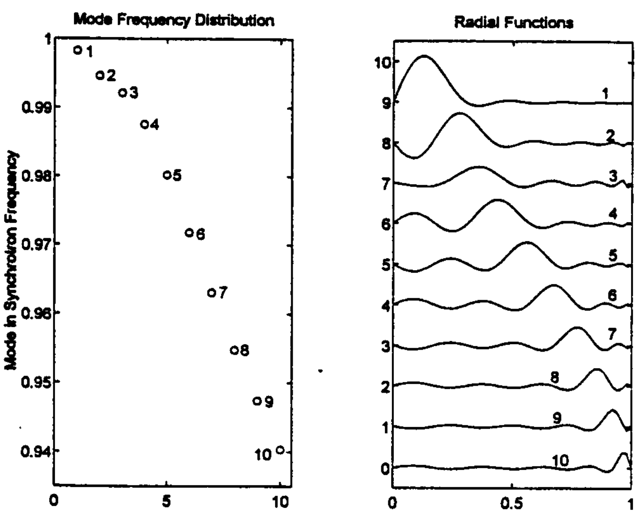

Fig. 2 Frequency Spread and Radial Modes

An example of the *m*=1 mode with RF non-linearity induced synchrotron frequency spread is shown in Fig. 2, where the coherent frequency distribution together with the corresponding radial functions at low intensity are shown. In Fig. 3 a comparison of the unnormalized perturbation line densities with and without frequency spread is shown. It can be shown that for a narrow-band impedance such as the RF cavity, an antidamping mode involves more the particles in the bunch center, and a damping mode involves the bunch edge [5], which can be observed through the line density variation. In a case of antidamping, the radial modes responsible for the instability gradually expand from the beam center to the edge, as the beam intensity is increased, which can be used to interpret the Landau damping.

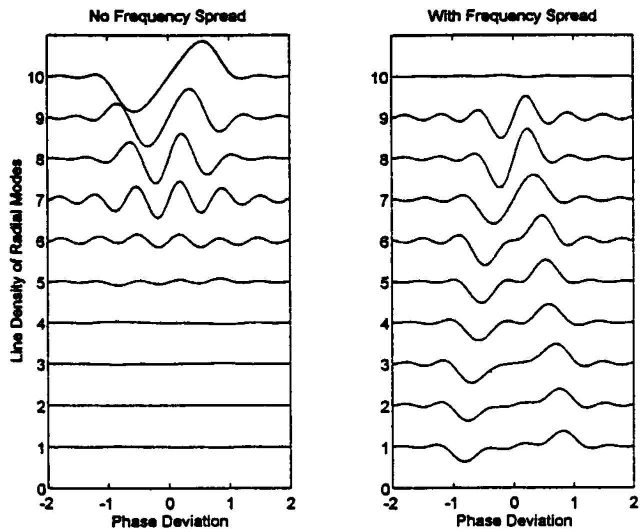

Fig. 3 Comparison of Line Densities

The effect of frequency spread itself generates no coherent distribution density evolution. As the beam intensity increases, the feedback governed by the Vlasov equation becomes effective, and coherence is taking place. The approach used in (4) is to approximate the system by a number of subsystems. This approach is convenient for the computation, and also allows physical insight into Landau damping. For most impedances, such as resonators with not too high resonant frequencies, only a small dimension of expansion is needed to get a good result [5].

IV. MODE COUPLING

When mode coupling is considered, we can write,

$$\omega \begin{bmatrix} \alpha^{(m)} \\ \alpha^{(m')} \end{bmatrix} = \left[\begin{bmatrix} m\omega_S \mathrm{I} & 0 \\ 0 & m'\omega_S \mathrm{I} \end{bmatrix} + \begin{bmatrix} M^{(m,m)} & M^{(m,m')} \\ M^{(m',m)} & M^{(m',m')} \end{bmatrix}\right] \begin{bmatrix} \alpha^{(m)} \\ \alpha^{(m')} \end{bmatrix} \tag{5}$$

where only the coupling between the mode *m* and *m*' is included. The equation can, however, be easily extended to multi-mode coupling and to the case with frequency spread. Comparing (5) with (4), we find some similarities, such as that both are different from (1) with an intensity independent matrix in the system matrix. There are also differences such as that the matrix M in (5) is not symmetric as is the case in (4).

In general the orthogonality of the orthogonal polynomials between different asimuthal modes is not necessarily guaranteed. To quality (5) as an eigenvalue problem, the rotation factors $e^{jm\theta}$ has to be included in the orthogonal basis. Therefore, the radial modes in a mode coupling situation are implicitly defined by the following distribution,

$$\psi_p^{(m)}(r,\theta) = \sum_{m'=-\infty}^{\infty} R^{(m,m')}(r) e^{jm'\theta} \tag{6}$$

We observe that the radial mode in a mode coupling lost its sole dependence on r, it, however, kept the most important characteristic, which is that the whole distribution pattern of a radial mode in the phase space bears an identical coherent frequency ω .

Consider coupling between the modes m=1 and m=2. As an example, as the intensity increases from 1 to 9, the line densities of the two responsible radial modes in the two azimuthal modes are shown in Fig. 4. The evolutions of these modes can be observed.

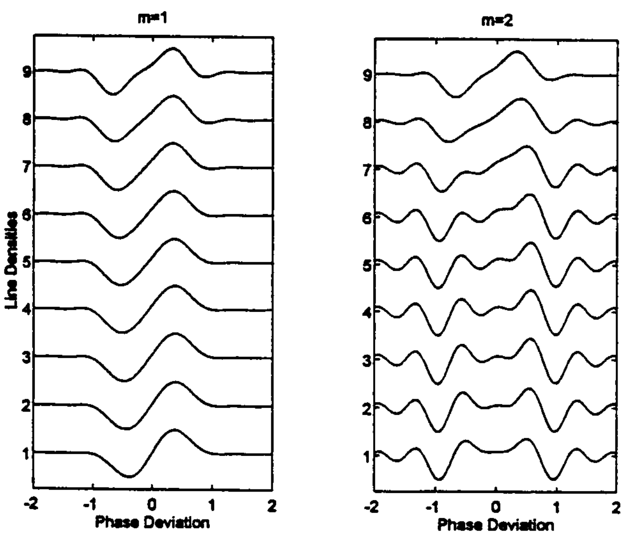

Fig. 4 Line Density Evolution for Mode Coupling

An interesting aspect of experimentally observed radial modes is reported in [6]. The observed modes are discrete, which agrees with the concept of the radial modes applied in this article. Also reported are some problems in observing these modes. We note that below the instability threshold often the radial modes responsible to the coupling are more difficult to observe, because usually before the coupling they are damped, not antidamped. Meanwhile there are other modes, which are less damped or not damped at all, therefore if the beam is excited, these modes are easily observed. They, however, have different coherent frequencies from the modes responsible for the instability thresholds. This can be shown by an example for the coupling between the m=1 and m=2 modes. The damping is especially strong for the m=±1 coupling, which is in fact the first Robinson criterion.

V. POTENTIAL WELL DISTORTION

With potential well distortion, two things happened. First, the stationary distribution may depend on not only r but also θ . For a Gaussian distribution, this problem can be easily treated. Second, the synchrotron frequency becomes dependent on several variables, including the stationary distribution, which itself is nonlinearly dependent on the beam intensity. Considering that the stable phase shift can be corrected by the RF phase feedback, and assuming that the distortion is not so severe that the original orthogonal polynomial expansion still can be used, the problem can be formulated similarly to equation to (4) as,

$$(\omega - m\omega_{SC})\alpha^{(m)} = \left(mG^{(m)} + M^{(m)}\right)\alpha^{(m)} \tag{7}$$

where $G^{(m)}$ can be calculated from the spectrum of the stationary distribution. Since the stationary distribution is distorted by the beam intensity induced feedback, this matrix is no longer constant, but non-linearly intensity dependent. It is, however, still real and symmetric. A radial mode evolution can be studied for this formulation.

In [7] a method using meshes in the radial direction as the orthogonal basis in the expansion is proposed, which has a potential to include perhaps more extended instability mechanisms. Although the general convergence is not considered as a problem if a large dimension expansion is used, the very concept of the radial modes is, however, buried in the large amount of eigenvectors, most of them represent in fact incoherent motions [8].

By using well established orthogonal polynomials for a non-distorted stationary distribution and using (7), the radial modes can be obtained and used to help the understanding of the mechanism of the instabilities. The convergence and the efficiency in using these polynomials with potential well distortion should be studied analytically and experimentally.

VI. REFERENCES

[1] F. Sacheren, CERN/SI-BR/72-5, 1972.

[2] G. Besnier, Nucl. Instru. Methods, Vol.164, p.235, 1979.

[3] B. Zotter, CERN SPS/81-(18-20), 1981.

[4] S.Y. Zhang, these proceedings.

[5] S.Y. Zhang and W.T. Weng, Hefei Workshop, China, July, 1994, to be published.

[6] D. Brandt, K. Cornelis, and A. Hofmann, EPAC, 1992.

[7] K. Oide and K. Yokoya, KEK Preprint 90-10, 1990.

[8] M. D'yachkov and R. Baartman, EPAC, 1994.