A MATHEMATICAL MODEL FOR INVESTIGATING CHROMATIC ELECTRON BEAM EXTRACTION FROM A PULSE STRETCHER RING.

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I. INTRODUCTION

A chromatic beam extraction from a pulse stretcher ring can be realized at a non-zero chromaticity of the machine. The finite chromaticity value stimulates the dependence of the betatron oscillation frequency on the particle energy. This permits one to carry out the beam extraction through the use of the betatron resonance by changing the particle energy.

Considering that during the chromatic extraction the particles are found in the vicinity of the RF separatrix or cross it, the computer code must ensure the simulation of the nonlinear motion of electrons inside and outside of the RF separatrix with due account of the synchrotron radiation effects.

II. A DIPOLE MAGNET WITH A UNIFORM FIELD.

In view of the radiations, the differential equation, the solution of which describes the electron dynamics in the dipole magnet, can be written as [1]:

$$mc(\frac{d^2\vec{r}}{ds^2}) = (\frac{e}{c})[(\frac{d\vec{r}}{ds})\times\vec{H} + \vec{f}(\frac{dt}{ds})] \tag{1}$$

where e, m are the electron charge and rest mass respectively, c is the velocity of light, $\vec{r}$ is the electron radius vector defining the particle position in the coordinate space, s is the parameter related to the laboratory time by $ds = cdt\left(1 - v^2 / c^2\right)^{1/2}$, v is the electron velocity, $\vec{f}$ is the force vector of the radiation reaction:

$$\vec{f} = -\frac{2e^4}{3m^2c^7}\left(\frac{d\vec{r}}{dt}\right)\left[\left(\frac{d\vec{r}}{dt}\right)\times\vec{H}\right]^2, \tag{2}$$

$\vec{H}$ is the magnetic field strength vector.

Under the assumption that $\vec{H}$ comprises only the axial component and is independent of the coordinates and time, eq.(1) can be integrated in the Cartensian system. It is convenient to present the integration results in the cylindrical frame (ρ, φ, z) as:

$$\frac{E}{\varepsilon} = \sqrt{\left.1 + c^2\left(1 - \frac{\varepsilon^2}{E^2}\right) \middle/ \left(2Bs + c_1\right)\left(v_{0\rho}^2 + v_{0\varphi}^2\right)\right.} \tag{3}$$

$$\frac{\left(v_\rho^2 + v_\varphi^2\right)}{c^2} = \frac{\left(\varepsilon / E_0\right)^2}{\left(2Bs + c_1\right)} \tag{4}$$

$$\left(\frac{v_z}{v_{0z}}\right) = \left[\frac{c_1}{\left(2Bs + c_1\right)}\right]^{1/2} \tag{5}$$

$$z = \frac{v_{0z}\sqrt{c_1}\left(\frac{E_0}{\varepsilon}\right)\left[\left(2Bs + c_1\right)^{1/2} - c_1^{1/2}\right]}{Bc} + z_0 \tag{6}$$

$$\rho = \left[\rho_0^2 + \left(\frac{\pi}{BD}\right)\left(\Phi_1^2 + \Phi_2^2\right)\right.$$

$$\left. + \rho_0\sqrt{\left(\pi / BD\right) \Big/ v_{0\rho}^2 + v_{0\varphi}^2}\left[v_{0\varphi}\Phi_1 + v_{0\rho}\Phi_2\right]\right]^{1/2} \tag{7}$$

$$v_\rho = \left\{\frac{c}{\rho}\left(\frac{\pi}{BD}\right)^{1/2}\left(\frac{\varepsilon^2}{E^2}\right)\left[\Phi_1\sin 2\Omega + \Phi_2\cos 2\Omega\right] + \right. \tag{8}$$

$$\left.\left(\frac{\varepsilon}{E}\right)\left(\frac{\rho_0}{\rho}\right)\left[\frac{c\left[v_{0\rho}\cos 2\Omega + v_{0\varphi}\sin 2\Omega\right]}{\left(v_{0\rho}^2 + v_{0\varphi}^2\right)^{1/2}}\right]\right\}\left(2Bs + c_1\right)^{-1/2}$$

$$\tan\varphi = \frac{M\left[v_{0\varphi}\Phi_2 - v_{0\rho}\Phi_1\right]}{M\left[v_{0\varphi}\Phi_1 - v_{0\rho}\Phi_2\right] + \rho_0} \tag{9}$$

$$t = \left(\frac{1}{2Bc}\right)\left\{\sqrt{\left(2Bs + c_1\right)\left(2Bs + L\right)} - \sqrt{Lc_1} + \right.$$

$$N\ln\left\{\frac{\left[4Bs + L + c_1 + 2\sqrt{\left(2Bs + L\right)\left(2Bs + c_1\right)}\right]}{\left(\sqrt{L} + \sqrt{c_1}\right)^2} \Big/ \right. \tag{10}$$

$$\left.\left.\frac{\left[4Bs + L + c_1 - 2\sqrt{\left(2Bs + L\right)\left(2Bs + c_1\right)}\right]}{\left(\sqrt{L} - \sqrt{c_1}\right)^2}\right\}\right\}$$

where

$$\Phi_1 = \left[s(u) - s(u_0)\right]\cos u_0 - \left[c(u) - c(u_0)\right]\sin u_0 ,$$

$$\Phi_2 = \left[c(u) - c(u_0)\right]\cos u_0 - \left[s(u) - s(u_0)\right]\sin u_0 ,$$

ρ, φ, z are the radius, azimutal angles and axial coordinate, $s(u), c(u)$ are the Fresnel sine and cosine integrals, respectively

$$c(u) = (2\pi)^{-1/2}\int_0^u \left(\frac{\cos\eta}{\sqrt{\eta}}\right) d\eta, s(u) = (2\pi)^{-1/2}\int_0^u \left(\frac{\sin\eta}{\sqrt{\eta}}\right) d\eta$$

$$u = (2Bs + c_1)\alpha \qquad u_0 = c_1\alpha$$

ε is the rest energy of the electron, E is its total energy,

$$v_\rho = \frac{d\rho}{dt}, v_\varphi = \rho\frac{d\varphi}{dt}, v_z = \frac{dz}{dt}, \Omega = \frac{Ds}{2}$$

$$B = \left(\frac{2e^4H^2}{3m^3c^6}\right), D = \frac{eH}{mc^2}, L = \frac{c^2}{\left(v_{0\rho}^2 + v_{0\varphi}^2\right)},$$

$$C_1 = L\left(\frac{\varepsilon^2}{E^2}\right), M = \frac{\left(\pi/BD\right)^{1/2}}{\left(v_{0\rho}^2 + v_{0\varphi}^2\right)^{1/2}},$$

$$\alpha = \left(\frac{D}{2B}\right), N = L\left[1 - \left(\frac{\varepsilon}{E_0}\right)^2\right]$$

The index '0' denotes that the associated quantities refer to the initial moment $t_0 = 0$, $\varphi_0 = 0$. Formulas (4)-(12) enable one to calculate all the necessary quantities.

The procedure of calculating the electron parameters at the exit of the dipole magnet with the angular length φ should be carried out in the following order.

First, the s value is obtained from formula (9) and is substituted into formulas (3-8) and (10). Simultaneously, $E, v_z, \left(v_\rho^2 + v_\varphi^2\right), z, \rho, t$ will be calculated.

Eventually, by the use of formula (4) and the obtained v_ρ value, the v_φ value is calculated.

Formulas (3-10) enable one to calculate all the necessary quantities. However, their employment in the given form is somewhat inconvenient for the computer. Therefore, to simplify the procedure, the following approximations should be made.

By virtue of the fact that the arguments of the Fresnel sine and cosine integrals have large values ($u_0 >> 4*10^{15}\gamma^{-2}H^{-1}$, is in gausses), the s and c can be calculated by using their asymptotic representations [2]:

$$S(u) = \left(1/2\right) - \frac{1}{\sqrt{2\pi u}}\cos u + 0\left(1/u^2\right)$$

$$C(u) = \left(1/2\right) - \frac{1}{\sqrt{2\pi u}}\sin u + 0\left(1/u^2\right)$$

On calculating t as a function of s, in expression (10) we use the condition of smallness of $2Bc/L$ and $2Bs/c$, as compared with the unity: $t \approx \left(s/2c\right)\left(E_0/\varepsilon\right)\left[5 - 3\left(\varepsilon/E_0\right)^2\right]$

III. CYLINDRICAL RF CAVITY.

For the cylindrical Rf cavity operating in the fundamental TM_{010} mode, the electrical and magnetic RF field components in the cylindrical coordinate frame can be represented as:

$$\varepsilon_z = -\varepsilon^0\sin(\omega t + \alpha)I_0(kr), H_\vartheta = -\varepsilon^0\cos(\omega t + \alpha)I_1(kr),$$

where I_0 and I_1 are the Bessel functions; $k = \frac{P}{a}$; P is the first root of the equation $I_0(Pa) = 0$; a is the cavity cylinder radius; r is the polar radius. The z axis is directed along the cavity axis; ϑ is the azimutal angle; $\omega = 2\pi\nu; \nu$ is the RF field frequency.

By solving the differential Newton -Lorents equation $\frac{dP}{dt} = e\varepsilon + \frac{e}{c}\left[\frac{d\vec{r}}{dt}\times\vec{H}\right]$ for the electron moving in the cavity, we can obtain:

$$Er^2\dot{\vartheta} = E_0 r_0^2\dot{\vartheta}_0 = C_1 \qquad (11)$$

$$E\dot{z} = \left(\frac{e\varepsilon^0 c^2}{\omega}\right)I_0(kr)\cos\omega t + C_2, \qquad (12)$$

$$C_2 = E_0 z_0 - \left(\frac{e\varepsilon^0 c^2}{\omega}\right)I_0(kr_0)\cos\alpha.$$

$$\frac{\dddot{\vartheta}}{\dot{\vartheta}} - \frac{3}{2}\left(\frac{\dddot{\vartheta}}{\dot{\vartheta}}\right)^2 + 2\dot{\vartheta}^2 - \frac{2ec\dot{z}H_\vartheta}{\left(EC_1/\dot{\vartheta}\right)^{1/2}} + \frac{\ddot{E}}{E} - \frac{1}{2}\left(\frac{\ddot{E}}{E}\right)^2 = 0 \qquad (13)$$

$$\dot{E} = e\dot{z}\varepsilon_z \qquad (14)$$

Assuming $kr\langle\langle 1$, we put $I_0(kr) \equiv 1$. Under this approximation, we have $I_1(kr) \equiv 0$, and therefore, in eq. (13) should be also set to equal zero.

Using eqs. (11)-(14), we come to the following formulas:

$$E \cong E_0 \left\{ 1 + \frac{2eC_2 s^0}{E_0^2 \omega} \left[\cos(\omega t + \alpha) - \cos\alpha \right] \right.$$

$$\left. + \frac{e^2 \varepsilon^{0^2} c^2}{2E_0^2 \omega^2} \left[\cos(2\omega t + 2\alpha) - \cos 2\alpha \right] \right\}^{1/2} \qquad (15)$$

$$\dot{z} \cong \frac{\left[\frac{e\varepsilon^0 c^2}{2E_0^2 \omega^2} \cos(\omega t + \alpha) + C_2 \right]}{E} \qquad (16)$$

$$z \cong \frac{\dot{z}_0 + e\varepsilon^0 c^2 \cos\alpha}{E_0 \omega \gamma_{z0}^2} t - \frac{e\varepsilon^0 c^2}{E_0 \omega \gamma_{z0}^2} \left[\sin(\omega t + \alpha) - \sin(\alpha) \right]$$

$$\dot{\vartheta} \cong \dot{\vartheta}_0 \Big/ \chi_1^2 - \frac{\ddot{\vartheta}_0}{\dot{\vartheta}_0} \chi_1 \chi_2 + \left[\left(\frac{\ddot{\vartheta}_0}{2\dot{\vartheta}_0^2} \right)^2 + 1 \right] \dot{\vartheta}_0^2 \chi_2^2 \qquad (17)$$

$$2\vartheta = \frac{\arctan\left[2\chi_2 \left(\dot{\vartheta}_0^{-1} \chi_1 - \frac{\ddot{\vartheta}_0}{2\dot{\vartheta}_0^2} \chi_2 \right) \right]}{\left(\dot{\vartheta}_0^{-1} \chi_1 - \frac{\ddot{\vartheta}_0}{2\dot{\vartheta}_0^2} \chi_2 \right)^2 - \chi_2^2} + 2\vartheta_0 \qquad (18)$$

$$\chi_1 = \left(\frac{E}{E_0} \right)^{1/2} \left(1 - \frac{1}{2} Y \right); \chi_2 = (EE_0)^{1/2} Y; (\chi_1 \dot{\chi}_2 - \dot{\chi}_1 \chi_2) = 1$$

$$Y = \frac{2}{E_0 \omega \sqrt{(1 - a\cos\alpha)^2 - a^2}} \arctan\left[\sqrt{\frac{1 - a\cos\alpha - a}{1 - a\cos\alpha + a}} \left(\tan\frac{\omega t + \alpha}{2} - \tan\frac{\alpha}{2} \right) \right],$$

where

$$\dot{\vartheta}_0 = \frac{v_{\vartheta 0}}{r_0}; \ddot{\vartheta}_0 = -\left(\frac{2v_{r0}}{r_0} \right) + \frac{e\dot{z}_0 \varepsilon^0 v_{\vartheta 0} \sin\alpha}{E_0 r_0}; \gamma_{z0}^2 = \left[1 - \left(\frac{\dot{z}_0}{c} \right)^2 \right]^{-1};$$

$$v_{r0} = \dot{r}_0 = \left(\frac{dr}{dt} \right)_0; a = \left(\frac{e\dot{z}_0 \varepsilon^0}{E_0 \omega} \right); t_0 = 0$$

By substituting the cavity length value equal to λ/2, where λ is the RF field wave length, into (16), we can calculate the particle transit time

$$\tau \approx \frac{\lambda}{2\dot{z}_0} \left[1 - \frac{ec\varepsilon^0 \cos\psi}{E_0 \omega \gamma_{z0}^2} - \frac{2ec\varepsilon^0 \sin\psi}{\pi E_0 \omega \gamma_{z0}^2} \right] \qquad (19)$$

It should be noted that at $c_1 = 0, \dot{\vartheta} \equiv 0$ the $\dot{r}$ and r values are given by

$$\dot{r} = \frac{\dot{r}_0 E_0}{E}, r = r_0 E_0 Y$$

In the general case, the values of $\dot{r} = v_r$ and r should be calculated by formulas

$$v_r = c\left[1 - \left(\frac{\varepsilon}{E} \right)^2 - \left(\frac{\dot{z}}{c} \right)^2 - \left(\frac{v_\vartheta}{c} \right)^2 \right]^{1/2} \qquad (20)$$

$$r = \left(\frac{c_1}{E\dot{\vartheta}} \right)^{1/2} \qquad (21)$$

The formulas presented here to calculate the dynamic variables after it traversed the cavity can also be used for the case, where ε^0 and ω are the slow functions of time: $\left(d\omega/dt \right) \Big/ \left(\omega/\tau \right) \langle\langle 1$, $\left(d\varepsilon^0/dt \right) \Big/ \left(\varepsilon^0/\tau \right) \langle\langle 1$, t is the particle transit time.

IV. Conclusion.

The algorithms described in [3,4] can be used to describe the dynamic characteristics of particles in the other magnetic elements of storage ring. The calculation time of the proposed mathematics model will be considerably longer. However it is supposed that the adequately of the given model to the physical process going at the extraction achromatic regime will compensate this shortcoming.

V. REFERENCES.

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