ANALYTICAL MODEL FOR EMITTANCE COMPENSATION IN RF PHOTO-INJECTORS

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In this paper we present a new model to represent analytically the transverse beam dynamics in RF photo-injectors. It consists basically of an enhanced Kim's model [1], with incorporation of RF ponderomotive focusing effects, external magnetic focusing and a perturbative treatment of space charge along the beam envelope. Applying the resulting formulas it is possible to predict with high accuracy the transverse beam envelope behaviour in a multi-cell RF gun, as well as the operating conditions to achieve space charge emittance compensation according to Carlsten's scheme [2]. The agreement with sophisticated numerical simulations is really quite satisfactory, as well as the match with experimental measurements of the predicted operating range for emittance compensation.

I. BEAM ENVELOPE IN MULTI-CELL RF GUNS

The basic model adopted for calculating the beam envelope consists actually of an indefinitely long RF gun, supporting a $TM_{010\text{-}\pi}$ standing resonant mode (frequency ν) with accelerating field $E_z = E_0\cos(kz)\sin(\omega t+\varphi_0)$ (E_0 is the peak field at the cathode, located at z=0, $k=\omega/c$, $\omega=2\pi\nu$). The field is expanded linearly off-axis to find the transverse E_r and B_φ components [1]. An external solenoid is assumed to be folded around the first 2+1/2 cells of the RF gun cavity, producing a constant magnetic field $B_z = B_0$ from z=0 up to $z_C=(5/4)\lambda$.

Under the approximation $\alpha > 1/2$, where $\alpha = eE_0/(2mc^2k)$ is the dimensionless field intensity, it has been extensively shown elsewhere [3] that the beam envelope conditions σ_2 and σ_2' at the second iris location $z_2=(3/4)\lambda$ can be written, for a gaussian charge density distribution in the bunch, as

$$\sigma_2 = \sigma_{cat}\left(1+\Delta_{SC}-\Delta_B\right)$$
$$\sigma'_{orb} = \frac{\gamma'}{\gamma_2}\sigma_{cat}\left[\Delta p^{RF}+\mu_{SC}-2\Delta_B\left(1+\Delta p^{RF}-\Delta_B\right)\right] \quad 1)$$
$$\sigma'_2 = \sigma'_{orb}-\frac{\gamma'}{2\gamma_2}\sigma_2$$

where σ_{cat} is the laser spot size at the cathode, Δp^{RF} is the RF defocusing kick

$$\Delta p^{RF} = 2-\frac{Log(\gamma_1)}{\gamma_1-1}-\frac{Log(\gamma_2)}{8}\left[2-\frac{Log(\gamma_1\sqrt[3]{\gamma_2})}{2(\gamma_2-1)}\right],$$

Δ_B is the magnetic focusing term $\Delta_B = (1/2)[k_s\mathrm{Log}(\gamma_2/\gamma_B)]^2$ ($k_s=cB_0/E_0$) and Δ_{SC} is the space charge defocusing kick $\Delta_{SC} = (1+\mu_{SC})[1-\mathrm{Log}(\gamma_2)/(\gamma_2-1)]$, calculated *a la Kim* . The space charge dependence on bunch and field parameters is specified by $\mu_{SC} = Z_0\, I\, \zeta\, /\, (8\gamma' E_0\, Ia\, \sigma^2_{cat})$, where Z_0=377 ohm, I is the bunch peak current, Ia=Alfven current, $\gamma' = \alpha k$ is the dimensionless energy gain per unit length and ζ specifies the dependence on the bunch aspect ratio $A = \sigma_r / \sigma_z$, $\zeta = 1 / (2.45+1.82A^{1.25}-0.55A^{1.5})$. The beam energies γ_1 and γ_2 (in rest mass units) at the first and second iris are given by $\gamma_1 = 1 + \alpha\pi/2$ and $\gamma_2 = 1 + 3\alpha\pi/2$ ($\gamma_B = 1 + \alpha\pi/4$) .

After the second iris the beam envelope can be easily tracked, as far as the space charge can be assumed negligible beyond this point, by applying the RF focusing transport matrix for relativistic beams [4] with initial conditions σ_2 and σ'_{orb} given by eqs.1 (σ_2' gives the *secular* orbit divergence and will be used later as initial condition for the envelope equation). The matrix gives at any position the beam spot σ of the average *secular* orbit and the beam divergence σ' of the actual orbit.

The agreement between analytically predicted envelopes and numerical simulation results are shown in Fig.1 for two 10+1/2 cell guns at typical frequencies and peak fields.

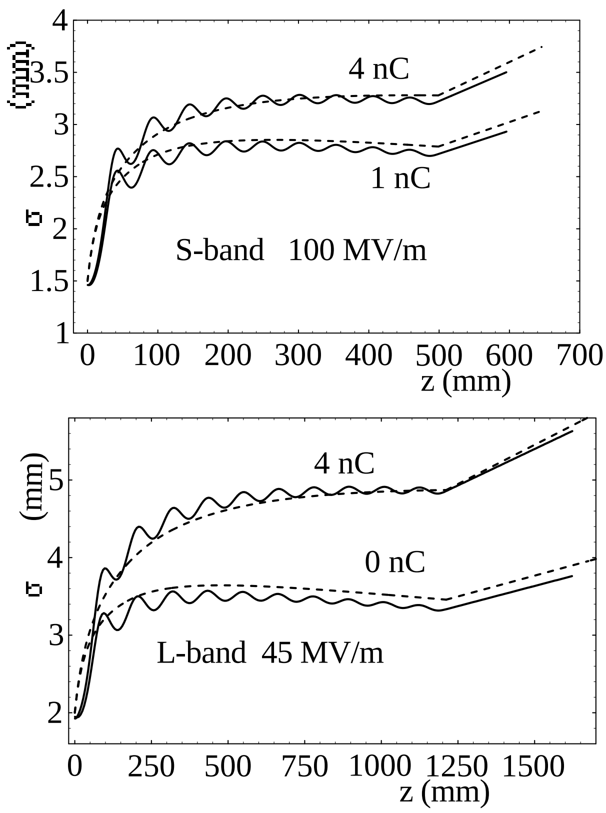

Figure 1: Beam envelopes through two different 10+1/2 cell RF guns (ν=2.856 GHz upper diagram, ν=1.3 GHz lower diagram). Dashed lines give the secular orbits analytically predicted, while solid lines are numerical simulation results.

In the upper diagram the bunch aspect ratio is A=1.25 with σ_{cat}=1.5 mm, corresponding to a peak current I=100 A at 1 nC and 400 A at 4 nC (z_2=79 mm , γ_2=8.7 , α=1.64); in the lower diagram A=0.83 with σ_{ca}ike problemsz_2=174 mm , γ_2=8.6 , α=1.62). The simulations were performed with the codes ATRAP [5] for the S-band gun and ITACA [6] for the L-band one. In this way the beam envelope can be predicted just on the basis of six main free parameters: three of them characterize the external fields, namely E_0, ν, B_0, while the bunch characteristics, determining the collective field, are specified by σ_{cat}, A, and I.

Eqs.1 are derived under the basic assumption of self-similar expansion of the charge density distribution, which stays gaussian in the (r,z) space under the effect of all the external as well as the collective forces acting on the bunch particle. The average bunch phase $<\varphi>$ is assumed to be $<\varphi>=\pi/2$, corresponding to maximum acceleration in the gun.

In case the magnetic focusing is not applied (B_0=0, as for Fig.1), the beam envelope behaviour is such that the space charge effects can be considered negligible after the second iris. In presence of an external magnetic focusing we need a different treatment of the beam dynamics. The envelope equation for a relativistic beam [7] seems the best approach:

$$\sigma'' + \sigma'\left(\frac{\gamma'}{\gamma}\right) + K_r \cdot \sigma - 2\frac{(I/Ia)}{\sigma \cdot \gamma^3} - \frac{\varepsilon_n^2}{\sigma^3\gamma^2} = 0 \qquad 2)$$

where $\sigma' = d\sigma/dz$, ε_n is the rms normalized emittance and K_r is the RF focusing gradient [4] $K_r = (k_s^2+1/8)(\gamma'/\gamma)^2$, which incorporates the contribution (k_s) from the magnetic field. This equation actually holds for an un-bunched beam: since the bunch aspect ratio in its rest frame, given by A/γ, can be considered small enough beyond the second iris, where typically $\gamma>5$, we assume eq.2 can be taken as a good approximation. Moreover, since we are interested in studying the conditions which give rise to space charge emittance compensation, the hypothesis of beam laminarity will be set up, implying that the emittance term in eq.2 is negligible. This is equivalent to assume that the beam envelope will not go through any crossover from the cathode up to the gun exit.

Under these assumptions, we can apply a Cauchy transformation to eq.2, setting $y = \mathrm{Log}(\gamma/\gamma_2)$ (recalling that $\gamma = 1 + \alpha k \bullet z = 1 + \gamma' z$) and obtaining

$$\frac{d^2\sigma}{dy^2} + \Omega^2\sigma = \frac{S}{\sigma}e^{-y} \qquad 3)$$

with $\sigma = \sigma(y)$ and $S = 2I/(Ia\,\gamma'^2\,\gamma_2)$. We solve eq.3 following two different techniques, according to two different domains: the first one is defined by $z_2 < z < z_c$ ($0<y<y_c$) and it is characterized by the focusing from the magnetic field, so that $\Omega^2 = 1/8 + k_s^2$. The 2nd one is defined by $z>z_c$ ($y_c=\mathrm{Log}((1+5\pi\alpha/2)/\gamma_2)$, $z_c=5\lambda/4$), hence $\Omega^2 = 1/8$. In the first domain the beam size σ is varying slightly with respect to σ_2, allowing to assume $\sigma=\sigma_2$ in the non linear term on the r.h.s. of eq.2. The general solution σ_I of the linearized equation becomes

$$\sigma_I = \sigma_2\frac{Cos\Omega y}{\sqrt{S}} + \dot{\sigma}_2\frac{Sin\Omega y}{\Omega\sqrt{S}} + \frac{\left[e^{-y} - Cos\Omega y + \frac{Sin\Omega y}{\Omega}\right]}{\sigma_2(1+\Omega^2)/\sqrt{S}} \qquad 4)$$

where $\dot{\sigma} = d\sigma/dy$ and $\dot{\sigma}_2 = \sigma_2'\,\gamma_2/\gamma'$.

Setting $\sigma_c=\sigma_I\,(y=y_c)$ and $\dot{\sigma}_c=\dot{\sigma}_I\,(y=y_c)$, we can solve perturbatively eq.3 in the second domain, assuming that the non linear term on the r.h.s. may be represented by a particular solution of the form 4). The perturbative solution σ_{II} becomes

$$\sigma_{II} = \left\{\left[\sigma_c - \frac{Se^{-y_c}}{\sigma_c\Psi}\right]Cos\left(\frac{y-y_c}{\sqrt{8}}\right) + \frac{Se^{-y+(y_c-y)\dot{\sigma}_c/\sigma_c}}{\sigma_c\Psi} + \left[\dot{\sigma}_c - \frac{Se^{-y_c}(1+\dot{\sigma}_c/\sigma_c)}{\sigma_c\Psi}\right]Sin\left(\frac{y-y_c}{\sqrt{8}}\right)\right\}\sqrt{S} \qquad 5)$$

where $\Psi = 1/8 + (1+\dot{\sigma}_c/\sigma_c)^2$. Eq.5 produces envelopes as those shown in Fig.2, for the same bunch and field parameters of Fig.1 (lower diagram, L-band gun), at various values for B_0. One should note that an angular kick $\Delta\sigma' = +\gamma'/2\gamma$ [4] must be added to the secular envelope at the gun exit in order to transform it back into the actual envelope.

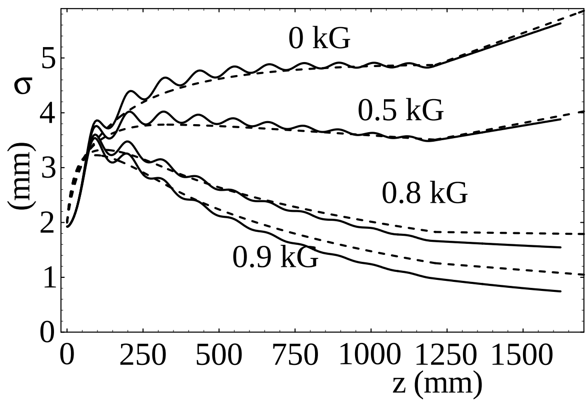

Figure 2: Beam envelopes through a 10+1/2 cell L-band RF gun (E_0=45 MV/m, I=200 A, Q=4 nC).

II. UNIVERSAL SCALING

Due to the excellent agreement between the analytical treatment and the numerical data, we believe that is possible to extract from the envelope expressions, eqs.4,5 and 1, useful informations on beam quality and RF gun performances. First of all, let us normalize the envelope eq. 3, which, in Cauchy space (σ,y), reads, for $y>y_c$, $\ddot{\sigma} + \sigma/8 = Se^{-y}/\sigma$, in order to reduce all the parameters to dimensionless quantities.

By defining the dimensionless quantity $\tau \equiv \sigma/\sqrt{S}$, the envelope eq. in the Cauchy dimensionless space (τ,y) reads

$$\frac{d^2\tau}{dy^2} + \frac{\tau}{8} = \frac{e^{-y}}{\tau} \qquad 6)$$

which is a universal scaled equation, independent on any external parameter. It is interesting to note that the function τ can be expressed as $\tau = \gamma' \bullet \lambda_p\sqrt{\gamma_2/\gamma}$: under this form it is clearly shown that τ scales like the ratio between the plasma wavelength $\lambda_p=2\pi c/\omega_p$ and the energy gain length $L_g = 1/\gamma'$. Moreover, a particular exact solution of equation 6) can be found to be $\tau^* = \sqrt{8/3}\,e^{-y/2}$: this solution is characterized by a constant ratio between the two fundamental scale lengths, L_g and λ_p, i.e. $\lambda_p/L_g = \sqrt{8/3}$. The corresponding beam size σ^* comes out to scale like $1/\sqrt{\gamma}$, namely $\sigma^* = \sqrt{8S\gamma_2/(3\gamma)}$, but, more relevant, τ^* is the only solution displaying a constant phase space angle δ, which is independent on initial conditions σ_c and $\dot\sigma_c$ in all of the three spaces (Cauchy dimensionless, Cauchy, real) . In fact, $\delta^* = \dot\tau^* / \tau = \dot\sigma^* / \sigma = \gamma\sigma'^* / (\sigma\gamma') = -1/2$, so that in both Cauchy spaces the phase space angle is a universal constant. The most relevant consequence is that δ, on this particular envelope σ^*, which will be called the *invariant envelope*, does not depend on the beam current (whose dependence is embedded in the expressions for σ_c and $\dot\sigma_c$): this is exactly the basic condition to get a vanishing linear correlated emittance. In fact, it is well known that the emittance growth from linear space charge effects is due basically to the spread in phase space distribution of different bunch slices, which get different kicks from the space charge field: these may be thought to be represented by different current amplitudes in the envelope eq.2 .

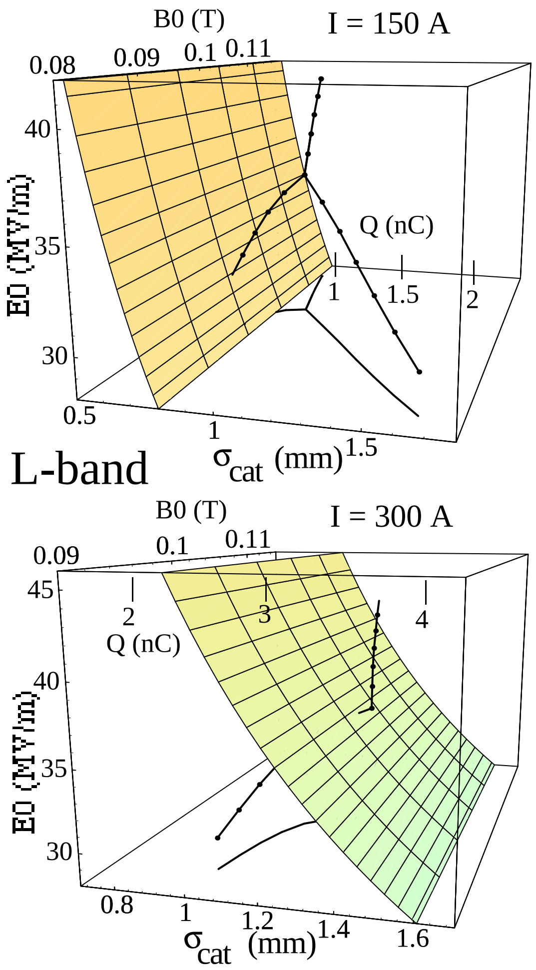

Figure 3: Solutions of eq.7 at ν=1.3 GHz, A=1, I=150 (upper diagram) and I=300 A (lower diagram), represented by the solid-dotted lines in 3-D sub-space (E_0, B_0, σ_{cat}). Since the bunch aspect ratio is kept constant, the displayed bunch charges are given by $Q=\sqrt{2\pi}\, I\bullet\sigma_{cat} / A$.

III. RF GUN OPERATING CONDITIONS FOR EMITTANCE COMPENSATION

In order to find the operating points, as functions of the six free parameters, we must solve the equation:

$$[\sigma_c - \sqrt{8S\gamma_2/(3\gamma_c)}\,]^2 + [\dot\sigma_c + (1/2)\sqrt{8S\gamma_2/(3\gamma_c)}\,]^2 = 0 \qquad 7)$$

whose solutions, in the 6-D space (ν, I, A, E_0, B_0, σ_{cat}), assure that σ_c and $\dot\sigma_c$ match the initial conditions of the invariant envelope. In this way the beam is transported from the cathode up to the gun exit with no space charge correlation, hence the emittance compensation is achieved.

In order to simplify the search for the solutions, we fix three parameters, namely ν, I and the aspect ratio A. The roots found in the 3-D sub-space (E_0, B_0, σ_{cat}) are plotted in Fig.3 for a L-band gun at I=150 A and I=300 A. The projections on the plane (B_0, σ_{cat}) are also plotted in the figure (solid lines), while the shaded surfaces set the limit of maximum charge extractable from the cathode (only points on the right of the surface are allowed).

Selecting one possible solution from Fig.3, namely ν=1.3 GHz, A=1, I=150 A, E_0=35 MV/m, B_0=1.02 kG and σ_{cat}=0.84 mm (so that Q=1 nC), we plot the corresponding envelope in Fig.4, for the nominal 150 A current and for lower and higher currents, 110,130,170,190 A, respectively: it is worthwhile to note that envelopes corresponding to currents different from the nominal one spread out in the first cells (see enclosed box), where the focusing action of the solenoid is dominant, but converge down into a common *invariant envelope* along the second domain (z_c>330 mm) of eq.6.

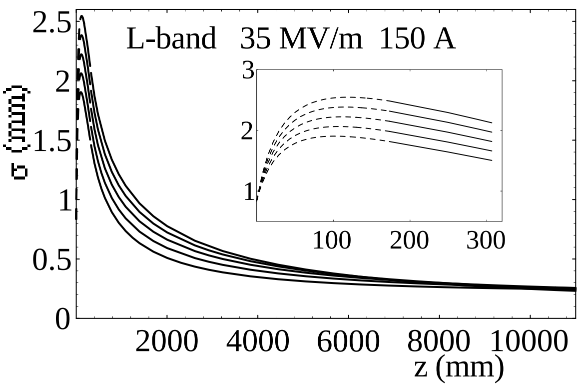

Figure 4: Envelopes corresponding to one of the solutions shown in Fig.3, for various currents around the nominal one.

IV. REFERENCES

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