EXPLICIT SOFT FRINGE MAPS OF A QUADRUPOLE

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Abstract

The analytic linear and nonliner zero-length fringe maps are calculated explicitly for a normal quadrupole up to 6-th order generators. A very simple leading term is found for the linear fringe map. The 4-th order leading term is the L. Whiting's hard edge result as expected. No significant leading term at the 6-th order are found. Our results should be useful to estimate the significancy of the soft fringe of a quadrupole to beam dynamics. The method used in this calculation can be used to compute the soft fringe maps of various magnets.

I. INTRODUCTION

Fringe fields exist for all kinds of magnets. They are an important source of nonlinearity in beam dynamics. Although maps including fringe fields can be calculated numerically using program like MARYLIE [3], [6], they are often neglected due to either their weak effects or the difficulties to handle them. It should be useful to have handy analytic fringe maps to estimate the significancy of the fringe fields at various orders. An analytic map will be physically more meaningful also. There are some analytic results for hard-edge fringe maps[4], [10] but none for soft-fringe maps. In this paper, we calculate the linear and nonlinear symplectic maps of a normal quadrupole fringe field. For simplicity, we consider only the geometric nonlinearity. The authors would like to thank A. Dragt for calculating fringe maps numerically using his MARYLIE.

II. POSITION DEPENDENT HAMILTONIAN

For an on-momentum particle in a normal quadrupole, the Hamiltonian reads

$$\begin{aligned} &H(x, P_x; y, P_y; z, \delta) \\ &= -\frac{eA_z}{P_0} - \left[1 - (P_x - \frac{eA_x}{P_0})^2 - (P_y - \frac{eA_y}{P_0})^2\right]^{\frac{1}{2}} \\ &\simeq -\frac{eA_z}{P_0} + \frac{1}{2}\left[(P_x - \frac{eA_x}{P_0})^2 + (P_y - \frac{eA_y}{P_0})^2\right] \end{aligned} \tag{1}$$

where P_x,P_y are the normalized momenta with respect to P_0, and $\delta = \Delta P/P_0$; $\vec{A}$ is the vector potential of the magnetic field. In the last step we dropped the kinematic nonlinear terms also. In fact, even the kinematic nonlinearity is significant for the whole quadrupole, it may not be important (at least at the 4-th order) in the fringe map we are considering because of the shortness of the fringe region.

For a magnetic field with cylindrical symmetry, its vector potential can be obtained from the on-axis field [5], [9]. In a conveniently chosen gauge, the non-zero components of a normal multipole are

$$A_r = \frac{\cos m\theta}{m} r \frac{\partial}{\partial z}\phi_m, \quad A_z = -\frac{\cos m\theta}{m} r \frac{\partial}{\partial r}\phi_m \tag{2}$$

where

$$\phi_m(r, z) = \sum_{l=0}^{\infty} \frac{(-1)^l\, m!}{2^{2l} l!\,(m+l)!} C_m^{(2l)}\, r^{m+2l}.$$

For a normal quadrupole with an on-axis gradient $G(s)$, $m = 2$ and $C_2 = \frac{1}{2}G(s)$. From the expansion of the vector potential, we can expand the Hamiltonian keeping up to 6-th order terms. This yields the approximate position dependent Hamiltonian to be used in our map calculation

$$\begin{aligned} H(s) =& \frac{1}{2}(P_x^2 + P_y^2) + \frac{1}{2}k(s)(x^2 - y^2) - \\ & \frac{1}{4}k'(s)(xP_x + yP_y)(x^2 - y^2) - \frac{1}{12}k''(s)(x^4 - y^4) \\ & + \frac{1}{32}k'^2(s)(x^4 - y^4)(x^2 - y^2) \\ & + \frac{1}{48}k'''(s)(xP_x + yP_y)(x^4 - y^4) \\ & + \frac{1}{256}k^{(4)}(s)(x^4 - y^4)(x^2 + y^2) + O(X^8) \end{aligned} \tag{3}$$

where $k(s) = \frac{e}{P_0}G(s)$ is the position dependent quadrupole strength.

If the quadrupole is long compared to its fringe field region, its strength approaches a constant $k_0 \equiv k(0)$ inside the quadrupole. We will treat the fringe field part as perturbation, i.e.

$$H(s) = H_0(s) + \tilde{H}(s) \longrightarrow H_0(s) \quad \text{outside fringe} \tag{4}$$

The perturbation term is chosen in two ways. In the nonlinear fringe map section, we use

$$\begin{aligned} H_0(s) &= \frac{1}{2}(P_x^2 + P_y^2) + \frac{1}{2}k(s)(x^2 - y^2) \\ \tilde{H}(s) &= H(s) - H_0(s) = \text{nonlinear terms in Eq.(3)} \end{aligned} \tag{5}$$

In the linear fringe map section, we choose $H_0(s)$ a piece-wise constant Hamiltonian of an ideal quadrupole

$$H_0(s) = \begin{cases} \frac{1}{2}(P_x^2 + P_y^2) + \frac{1}{2}k_0(x^2 - y^2) & s \le s_0 \\ \frac{1}{2}(P_x^2 + P_y^2) & s > s_0 \end{cases} \tag{6}$$

and $\tilde{H}(s) = \frac{1}{2}\tilde{k}(s)(x^2 - y^2)$, where

$$\tilde{k}(s) = \begin{cases} k(s) - k_0 & s \le s_0 \\ k(s) & s > s_0 \end{cases} \tag{7}$$

$s_0 = \frac{1}{k_0}\int_0^\infty k(s)ds$, the effective "magnetic length". Our results will be expressed in terms of the moments of $\tilde{k}(s)$

$$\begin{aligned} I_0 &= \int_{s_0}^\infty k(s)ds\ , \qquad I_1 = \int_{-\infty}^\infty \tilde{k}(s)(s - s_0)ds \\ I_2 &= \int_{s_0}^\infty k(s)(s - s_0)^2 ds\ , \quad I_3 = \int_{-\infty}^\infty \tilde{k}(s)(s - s_0)^3 ds \end{aligned} \tag{8}$$

and $\quad K = \int_{-\infty}^{\infty} k'(s)^2 ds \ , \qquad K_0 = \int_{-\infty}^{\infty} \tilde{k}(s)^2 ds$

$$K_1 = \int_{-\infty}^{\infty} k'(s)^2 (s - s_0)^2 ds$$

$$K_2 = \int_{s_0}^{\infty} d\, s \int_{s}^{\infty} d\, s' k(s) k(s')(s' - s) \qquad (9)$$

We assumed that $\tilde{k}(s)$ is anti-symmetric about the edge s_0. It is nature to characterize the fringe width via the rms width σ of the bell-shaped function $k'(s)$ as

$$\sigma^2 = \frac{\int_0^\infty k'(s)(s - s_0)^2 ds}{\int_0^\infty k'(s) ds} = \frac{2I_1}{k_0} \qquad (10)$$

3σ is a good measure of the half width of fringe region.

III. NON-LINEAR FRINGE MAP

Consider the map from the center of the quadrupole $s_1 = 0$ to a point s_2 that is far outside the fringe field region. Our goal is to find a simplectic fringe map $\mathcal{Q}_f$ which represents the fringe field effects so that the map $\mathcal{M}(s_1 \to s_2)$ can be written as

$$\mathcal{M}(s_1 \to s_2) = \mathcal{M}_Q(s_1 \to s_0)\ \mathcal{Q}_f\ \mathcal{M}_{drift}(s_0 \to s_2) \qquad (11)$$

where $\mathcal{M}_Q$ is the map of an ideal quadrupole of strength k_0 and length $s_0 = L_{eff}$; $\mathcal{M}_{drift}$ is the drift map from s_0 to s_2. (They may contain kinematic nonlinearity even though we neglect it in our calculation of $\mathcal{Q}_f$. This approximation may not be good at the 6-th order)

Before working on Eq.(11), we concentrate on the non-linear part, i.e. considering

$$\mathcal{M}(s_1 \to s_2) = \mathcal{R}_-(s_1 \to s_0)\ \tilde{\mathcal{Q}}_f\ \mathcal{R}_+(s_0 \to s_2) \qquad (12)$$

where $\mathcal{R}_\pm$ are exact linear maps. To calculate this, we choose the perturbation $\tilde{H}(s)$ as in Eq.(5), slice the time dependent Hamiltonians into pieces and move all the linear map before and after s_0 to the left and right side respectively using similarity transformation. This process is exact. Then we concatenate all the nonlinear pieces into a perturbation map $\tilde{\mathcal{Q}}_f$ via 2nd order BCH formula. Since we are concerned with up to 6-th order generators, 3rd and higher order BCH terms do not contribute in this case. Therefore

$$\tilde{\mathcal{Q}}_f = e^{:-\int_{s_1}^{s_2} ds \bar{H}(s) + \frac{1}{2}\int_{s_1}^{s_2} ds \int_{s}^{s_2} d\tilde{s}[\bar{H}(s), \bar{H}(\tilde{s})]:} \qquad (13)$$

where $\bar{H}(s) = \tilde{H}(s, R(s_0 \to s)X)$; X represents the phase space variables and $R(a \to b)$ is the exact linear matrix from a to b.

To carry out the integrations in Eq.(13), usually we need to know the exact linear matrix $R(s_0 \to s)$. However, since the fringe region is expected to be very short, we can Taylor expand $R(s_0 \to s)X$ about s_0 and truncate at a suitable order of $\Delta s = s - s_0$. The coefficients of the series involve the derivatives of $k(s)$ and the dynamical variables at location s_0, which can be obtained via the Hamiltonian equations. This is a unique point in our approach. It is also possible to get exact results via integrals involving sine-like and cosine-like orbits as was done in [10], [7].

There is a subtlety about the convergence of this approach. Although in principle it should converge because the linear orbit is a well behaved function, convergence could be slow due to cancellations among various orders. Fig.1 shows the deviation of the various order expansions in Δs from the exact linear orbit in the fringe region. We see that though the approximations inside the fringe region are getting better, they become worse outside. Fortunately, the fringe region is short. Therefore, the expansion needs to be good only within a certain window covering the fringe, and the integrations in Eq.(13) is over that window. This will suppress the high order moments and help convergence. The low order moments will not change much when the window is sufficiently large to maintain the orginal boundary conditions.

The 4-th order generators of $\tilde{\mathcal{Q}}_f$ result from the integration of Hamiltonian only. Expansion up to Δs^5 is used.

$$\begin{aligned} F_4 \simeq & \frac{k_0}{12}[P_x(x^3 + 3xy^2) - P_y(y^3 + 3yx^2)] \\ & - \frac{1}{72}[6k'(s_0)I_1 + k'''(s_0)I_3](x^4 + 6x^2y^2 + y^4) \\ & - \frac{k(s_0)I_1}{6}(5x^3P_x + 9xP_xy^2 + 9yP_yx^2 + 5y^3P_y) \\ & + \frac{I_1}{2}(xP_x + yP_y)(P_x^2 - P_y^2) \end{aligned} \qquad (14)$$

The first term is the well-known hard-edge result; in which case $I_1 = I_3 = 0$. The other terms are due to the soft edge; the correction is usually quite weak. However, the effect of the pseudo-octupole term on tune shift has been observed[8]. The coefficient of this term can also be given by a slightly more accurate form $(k_0 I_0 + K_0)/12$. A quick estimate of the coefficient I_1 can be done via Eq.(10).

The 6-th order generators of $\tilde{\mathcal{Q}}_f$ result from integrations of the 6-th order terms in $\tilde{H}$ (up to Δs^3) and the Poisson bracket(up to Δs^2) of the 4-th order terms in $\tilde{H}$

$$\begin{aligned} F_6 \simeq & -\frac{K}{288}(x^2 + y^2)(x^4 + 14x^2y^2 + y^4) \\ & + \frac{k_0 k'(s_0)}{384}(x^2 + y^2)(x^4 + 10x^2y^2 + y^4) \\ & - \frac{5K_1 k(s_0)}{96}(x^2 - y^2)(3x^4 + 2x^2y^2 + 3y^4) \\ & - \frac{K_0 - 2k_0 I_0}{8}(x^2 - y^2)^3 \\ & + \frac{k_0 k(s_0)}{48}(2P_x x^5 + 3P_y x^4 y + 11P_x x^3 y^2 + 11P_y x^2 y^3 \\ & \qquad + 3P_x x y^4 + 2P_y y^5) \\ & + \frac{5K_1}{96}(15P_x^2 x^4 - P_y^2 x^4 - 8P_x P_y x^3 y - 6P_x^2 x^2 y^2 \end{aligned}$$

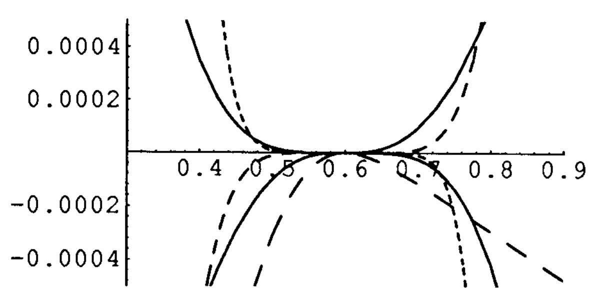

Figure. 1. Deviations of expansions from exact linear orbit

$$
\begin{aligned}
&-6P_y^2x^2y^2 - 8P_xP_yxy^3 - P_x^2y^4 + 15P_y^2y^4) \\
&-\frac{K_0 - 2k_0I_0}{24}(9P_x^2x^4 + 2P_y^2x^4 + 18P_xP_yx^3y + 3P_x^2x^2y^2 \\
&+3P_y^2x^2y^2 + 18P_xP_yxy^3 + 2P_x^2y^4 + 9P_y^2y^4) \\
&+\frac{k_0}{96}(-5P_x^3x^3 + 7P_xP_y^2x^3 - 3P_x^2P_yx^2y + 9P_y^3x^2y \\
&-9P_x^3xy^2 + 3P_xP_y^2xy^2 - 7P_x^2P_yy^3 + 5P_y^3y^3) \qquad (15)
\end{aligned}
$$

At 6-th order, there are no dominant term like the hard-edge term at 4-th order. The nonlinear fringe map in Eq.(12) is

$$\tilde{Q}_f = e^{:F:} \quad \text{with} \quad F = F_4 + F_6 \qquad (16)$$

IV. LINEAR FRINGE MAP

To obtain the fringe map Q_f in Eq.(11), we still need to work out the linear maps in Eq.(12). Now the Hamiltonians are given by Eq.(6). We use the same approach to treat the linear perturbation term and factor out a linear perturbation map at the edge s_0. Since we are dealing with 2-nd order terms, all terms in the BCH series are the same order. Therefore it may be necessary to sum an infinite series. However, the second order BCH formula yields very good approximation due to the weakness of fringe perturbation.

The two linear maps in Eq.(12) are found to be

$$
\begin{aligned}
\mathcal{R}_-(s_1 \to s_0) &= \mathcal{M}_Q(s_1 \to s_0)\, e^{:f_2^-:} \\
\mathcal{R}_+(s_0 \to s_2) &= e^{:f_2^+:}\, \mathcal{M}_{drift}(s_0 \to s_2) \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
\text{with} \quad f_2^- \simeq\ & \frac{I_0}{2}(x^2 - y^2) - \frac{I_1}{2}(xP_x - yP_y) + \frac{I_2}{2}(P_x^2 - P_y^2) \\
& -\frac{k_0I_2}{2}(x^2 + y^2) - \frac{k_0I_3}{3}(xP_x + yP_y) + \frac{K_2}{2}(x^2 + y^2) \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
\text{and} \quad f_2^+ \simeq\ & -\frac{I_0}{2}(x^2 - y^2) - \frac{I_1}{2}(xP_x - yP_y) \\
& -\frac{I_2}{2}(P_x^2 - P_y^2) + \frac{K_2}{2}(x^2 + y^2) \qquad (19)
\end{aligned}
$$

It is easy to concatenate the two linear fringe maps via 2-nd order BCH formula and get the total linear fringe map

$$\mathcal{R}_f = e^{:f_2^-:}e^{:f_2^+:} = e^{:f_2:} \qquad (20)$$

f_2 has a significant leading term generating the matrix $diag\{e^{I_1}, e^{-I_1}, e^{-I_1}, e^{I_1}\}$, which yields a scale change of the phase space. The order of this effect is given by the dimensionless parameter I_1.

To finish our calculation of the fringe map $\mathcal{Q}_f$, we combine Eqs.(11, 12, 16, 17) and obtain

$$\mathcal{Q}_f = e^{:f_2^-:}\, \tilde{Q}_f\, e^{:f_2^+:} = \mathcal{R}_f \exp\{: e^{:-f_2^+:} F :\} \qquad (21)$$

V. PEP-II $Q1$ MAGNET SOFT FRINGE FIELD EFFECTS

As an example, we will show the fringe field effects of the PEP-II $Q1$ magnet. It is a permanent magnet with the on-axis gradient $G(s)$ given by [1]

$$B_r \left[\frac{5}{8} z (\frac{v_1}{r_1^2} - \frac{v_2}{r_2^2}) + \frac{3}{8}\frac{1}{z}(\frac{1}{v_1} - \frac{1}{v_2}) + \frac{z}{8}(\frac{v_1^3}{r_1^2} - \frac{v_2^3}{r_2^2}) \right] \Bigg|_{z=s-L}^{z=s+L}$$

where $v_{1,2} = [1 + (\frac{z}{r_{1,2}})^2]^{-\frac{1}{2}}$. $s = 0$ is at the center of the quadrupole. L is half of its physical length. $r_{1,2}$ are the inner and outer radius. B_r is the remanent field of the permanent magnetic material. The parameters used are [2] $r_1 = 8.7$ cm, $r_2 = 16.6$ cm, $L = 60.0$ cm, $B_r = 1.05$ T, $G(0) = 10.64$ T/m. Also positron momentum $P_0 = 3.1$ GeV/C of the low energy ring is used. These yield the quantities defined in Eqs.(8,9) as

$$
\begin{aligned}
& k_0 = 1.029\text{m}^{-2},\ L_{eff} = 0.60\text{m},\ I_0 = 2.28 \times 10^{-2}\text{m}^{-1}, \\
& I_1 = 1.85 \times 10^{-3},\ I_2 = 7.56 \times 10^{-5}\text{m},\ I_3 = 1.95 \times 10^{-5}\text{m}^2, \\
& K = 5.65\text{m}^{-5},\ K_0 = 0.0127\text{m}^{-3},\ K_1 = 0.00678\text{m}^{-3}, \\
& K_2 = 1.05 \times 10^{-5}\text{m}^{-1}
\end{aligned}
$$

We calculated our fringe map generator coefficients and checked them againist MARYLIE, which shows agreement at 4-th order and terms of 6-th order with momentum power less than 2(e.g. xP_xy^4, x^6). The disagreement may be due to the kinematic terms. The four coefficients in F_4 is 0.086, 4.4×10^{-4}, -1.5×10^{-4}, 8.6×10^{-4}. For beam dynamics, more useful figures are the coefficients in the normalized coordinates ($x = \sqrt{\beta}\hat{x}$, $P_x = (\hat{P}_x - \alpha\hat{x})/\sqrt{\beta}$). At $Q1$, $\beta \simeq 100$ and $\alpha \simeq 40$. The coefficients read 340, 4.4, 0.6, 0.3×10^{-3}. The largest normalized coefficient in F_6 is the order of 2×10^4.

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