Arbitrary Order Transfer Maps for RF Cavities*

Johannes van Zeijts CEBAF, 12000 Jefferson Avenue, Newport News , VA 23606

Abstract

Current modeling of transfer maps for superconducting RF cavities at CEBAF includes only linear effects [1], [2]. Here we extend the transfer mapping modeling capability to include arbitrary order field information generated from the MAFIA field data. We include coupler kicks, normal and skew quadrupole focussing and higher order effects.

I. Introduction

Use of a Hamiltonian and associated conjugate variables is a prerequisite in the correct treatment of higher order effects and ensuring that the resulting transfer maps are symplectic. Much effort has been spent on creating realistic transfer maps for different magnets in the fixed energy case [3], [5], [6]. Here we present an extension of this work to the case where the design energy changes due to RF cavities.

The first part of this paper gives an introduction to the deviation variables and Hamiltonian used in the construction of transfer maps in the energy dependent case. The second part of the paper introduces the fields and the transfer map calculations for RF cavities. Finally we show how to derive transfer maps for elements that change the design energy, and give some examples for the CEBAF cavities.

II. Hamiltonian in Deviation Variables

Using standard techniques [4], [5] we derive the Hamiltonian in cartesian coordinates (x, y, t) with z as the independent variable as

$$K = -qA_z - \sqrt{\frac{pt^2}{c^2} - m^2c^2 - ((p_x - qA_x)^2 + (p_y - qA_y)^2)} \; . \tag{1}$$

We introduce deviation variables wrt. the design orbit and the (z-dependent) design energy:

$$p_t^1(z) = -\sqrt{m^2c^4 + p_1^2(z)c^2} = -\gamma(z)mc^2 \; , \tag{2}$$

where $p_1(z) = \beta(z)\gamma(z)mc$ is the design momentum, and wrt. the time of flight of the design orbit:

$$t_1(z) = \int \frac{1}{\beta(z)c} dz \; . \tag{3}$$

The scaling variables are wrt. a fixed momentum p_0, usually chosen to be the final momentum. This leads to the transverse deviation variables:

$$X \; , \; P_x = \frac{p_x}{p_0} \; , \; Y \; , \; P_y = \frac{p_y}{p_0} \; , \tag{4}$$

and for the temporal phase space deviation variables:

$$\tau = c(t - t_1(z)) \; , \; P_\tau = \frac{pt - p_t^1(z)}{p_0 c} \; . \tag{5}$$

Using this the Hamiltonian with arbitrary vector potential entries becomes, using $B_\rho^1 = p_1/q$ and $f = \frac{\beta(z)\gamma(z)}{\beta_0\gamma_0}$

$$\begin{aligned} H &= \frac{\gamma}{\beta_0\beta\gamma_0} - \frac{\tau}{\beta_0\gamma_0}\frac{\partial\gamma}{\partial z} - \frac{P_\tau}{\beta} - \frac{A_z}{B_\rho^0} \\ &- f\sqrt{1 - \frac{2P_\tau}{f\beta} + \frac{P_\tau^2}{f^2} - ((\frac{P_x}{f} - \frac{A_x}{B_\rho^1})^2 + (\frac{P_y}{f} - \frac{A_y}{B_\rho^1})^2)} \; . \end{aligned} \tag{6}$$

Arbitrarily complicated magnetic fields can be handled by inserting the appropriate vector potentials. From Hamilton's equation we have, assuming $A_x(0, z, t) = A_y(0, z, t) = 0$

$$\dot{P}_\tau|_{\text{design}} = \frac{\partial H}{\partial \tau}|_{\text{design}} \equiv 0 \Rightarrow \frac{\partial\gamma}{\partial z} = -\frac{q}{mc}\frac{\partial A_z(0, z, t)}{\partial \tau} \; , \tag{7}$$

which gives the equation for the energy increase.

III. Simple RF Fields

Here we consider standing-wave fields in a cavity with axial symmetry i.e. we only get the angle independent TM modes. following [7] an appropriate expansion for E_z is

$$E_z(r, z, t) = \sum_{n=1}^{N} a_n I_0(k_n r) \cos\left((2n - n_0)\frac{\pi z}{2d}\right) \sin(\omega t + \phi_0) \; , \tag{8}$$

with

$$k_n^2 = \beta_n^2 - \left(\frac{\omega}{c}\right)^2 = \left((2n - n_0)\frac{\pi}{2d}\right)^2 - \left(\frac{\omega}{c}\right)^2 \; . \tag{9}$$

The condition $n_0 = 1$ corresponds to the condition that $E_z(r, d, t) = 0$. An equivalent expression for the time independent vector potential $A(r, z, t) = A(r, z)\cos(\omega t + \phi_0)$ is:

$$A_z(r, z) = \frac{1}{\omega}\sum_{n=1}^{N} a_n I_0(k_n r)\cos(2n - n_0)\frac{\pi z}{2d} \; , \tag{10}$$

and

$$A_r(r, z) = \frac{1}{4df}\sum_{n=1}^{N}\frac{a_n}{k_n}(2n - n_0) I_1(k_n r)\sin(2n - n_0)\frac{\pi z}{2d} \; . \tag{11}$$

The energy gain of the reference particle is given by the differential equation

$$\frac{\partial\gamma}{\partial z} = \frac{q}{mc^2} E_0(z)\sin(\omega t_1(z) + \phi_0) \; , \tag{12}$$

where $E_0(z)$ is the on axis gradient given by

$$E_0(z) = \vec{}_0 \sum_{n=1}^{N} a_n \cos\left((2n - n_0)\frac{\pi z}{2d}\right) \; . \tag{13}$$

*This work was supported by the U.S. Department of Energy, under contract No. DE-AC05-84ER40150.

In practice RF cavities are phased to achieve maximum energy gain for a given field amplitude. In our approach a working point is found by using a two-dimensional Newton method where we vary the field amplitude and the phase to find a given energy increase which is to first order independent of the phase.

IV. RF Multipole Fields

Following standard techniques [8] here we consider the general RF field expansion inside a cylinder. TM and TE mode multipole fields starting with dipole terms are handled. The terms $m = 0$ correspond to the fields in the previous section, the term $m = 1$ correspond to a dipole field, the terms $m = 2$ correspond to quadrupole fields, etc. Since cavities are no longer assumed to be mirror symmetric we have to use both $\cos(\beta_n z)$ and $\sin(\beta_n z)$ terms in the Fourier series expansion. In this case we need a total of 8 independent terms per mode (n, m) to describe the fields adequately.

A. TM and TE modes

An appropriate expansion for E_z becomes

$$E_z(r,\theta,z)_{nm} = I_m(k_n r) E_z(\theta,z)_{nm} \tag{14}$$

where

$$\begin{aligned} E_z(\theta,z)_{nm} &= \cos(\beta_n z)\left(A^{TM}_{nm}\sin(m\theta) + B^{TM}_{nm}\cos(m\theta)\right) \\ &+ \sin(\beta_n z)\left(C^{TM}_{nm}\sin(m\theta) + D^{TM}_{nm}\cos(m\theta)\right) \end{aligned}$$

The TE modes are derived from

$$B_z(r,\theta,z)_{nm} = \frac{1}{c} I_m(k_n r) B_z(\theta,z)_{nm} \tag{15}$$

where

$$\begin{aligned} B_z(\theta,z)_{nm} &= \cos(\beta_n z)\left(A^{TE}_{nm}\sin(m\theta) + B^{TE}_{nm}\cos(m\theta)\right) \\ &+ \sin(\beta_n z)\left(C^{TE}_{nm}\sin(m\theta) + D^{TE}_{nm}\cos(m\theta)\right) \end{aligned}$$

We omit the full expansions for the fields $E^{TM}_{\theta}(r,\theta,z)_{nm}$, $E^{TM}_{r}(r,\theta,z)_{nm}$, $E^{TE}_{\theta}(r,\theta,z)_{nm}$ and $E^{TE}_{r}(r,\theta,z)_{nm}$, they are given in [9].

B. Vector potentials

The vector potentials become

$$A_z(r,\theta,z) = \frac{1}{\omega}\sum_{n,m} E_z(r,\theta,z)_{nm}\ , \tag{16}$$

and

$$A_\theta(r,\theta,z) = \frac{1}{\omega}\sum_{n,m}\left(E^{TM}_{\theta}(r,\theta,z)_{nm} + E^{TE}_{\theta}(r,\theta,z)_{nm}\right)\ , \tag{17}$$

and

$$A_r(r,\theta,z) = \frac{1}{\omega}\sum_{n,m}\left(E^{TM}_{r}(r,\theta,z)_{nm} + E^{TE}_{r}(r,\theta,z)_{nm}\right)\ . \tag{18}$$

V. Transfer Maps for RF Cavities

In this section we describe how to obtain the arbitrary order transfer map for RF cavities in the deviation variables

$$x,\ P_x,\ y,\ P_y,\ \tau,\ P_\tau\ . \tag{19}$$

Given the requested energy increase and phase set point, the transfer map will be derived as a Taylor series and transformed into Lie algebraic form for use in lattice design and tracking codes.

We start by expanding all fields in cartesian coordinates

$$r \to \sqrt{x^2+y^2}\ , \tag{20}$$

and

$$\omega t \to \frac{2\pi\tau}{\lambda} + \omega t_1(z)\ . \tag{21}$$

The vector potential in cartesian coordinates is

$$A_x = \frac{x A_r - y A_\theta}{\sqrt{x^2+y^2}}\ , \tag{22}$$

and

$$A_y = \frac{y A_r + x A_\theta}{\sqrt{x^2+y^2}}\ . \tag{23}$$

For general m the expansion of the Bessel functions is

$$I_m(kr) = r^m\left(\frac{1}{2}\right)^m k^m\left[\frac{1}{m!} + \frac{k^2 r^2}{4\ (1+m)!} + \cdots\right] \tag{24}$$

The expansion of the angular parts are

$$\sin(m\theta) = \frac{\Im(x+Iy)^m}{r^m}\ , \tag{25}$$

and

$$\cos(m\theta) = \frac{\Re(x+Iy)^m}{r^m}\ . \tag{26}$$

The terms r^m cancel in the product of these expansions as expected. The time dependent part is expanded in τ. Finally the expansions are inserted in the Hamiltonian (6), which is expanded analytically in the deviation variables to high order, using a symbolic manipulation program. The procedure to generate a transfer map from a Hamiltonian is described in [10], this basically involves integrating a set of ordinary differential equations for the coefficients of the Taylor series expansion. The system of ode's is extended by one for integrating $\gamma(z)$ and one for integrating $ct_1(z)$.

VI. CEBAF Cavities

As a first application of this approach we use the longitudinal electric field on axis for a typical CEBAF RF cavity. This field has been calculated using SuperFish and the coefficients in the expansion of $E_0(z)$ are presented in the table below. We have $\lambda = 0.2$ m, $f = 1497$ Mhz. The data is normalized to give a gradient of 1 MeV/m, and $d = 0.35$ m. Furthermore we present the data in Figure 1. In Figure 2 we plot the incremental energy increase (eq. 12) for the reference particle with the assumption that $\beta = 1$ and where we have chosen $\phi_0 = \pi/2$.

Work is in progress to apply the full power of the machinery presented in this paper to realistic cavities. For that purpose, the coefficients of the TM and TE modes are obtained by a Fourier transform of the MAFIA field data to high order [9].

n	a_n
1	0.2287767
2	-.5776482
3	0.7384962
4	1.396168
5	0.3219303
6	-.1126080
7	0.01861268
8	0.003110609
9	0.008141764
10	-.02498009
11	-.06243902
12	-.01799315
13	0.007256913
14	-.001471598

Table I
Expansion coefficients a_n for $E_0(z)$.

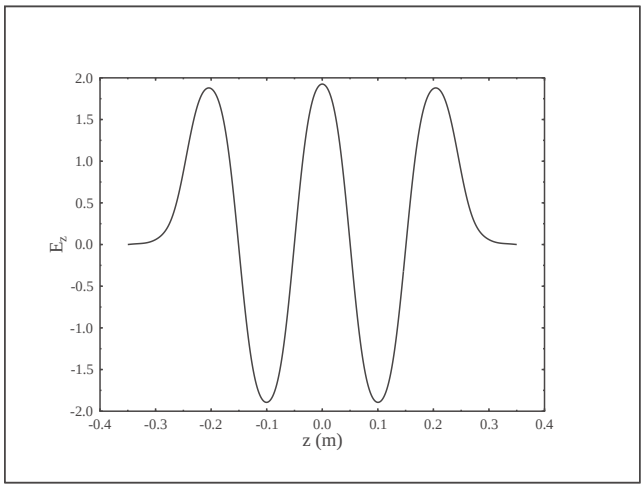

Figure. 1. Longitudinal electric field on axis for a CEBAF cavity.

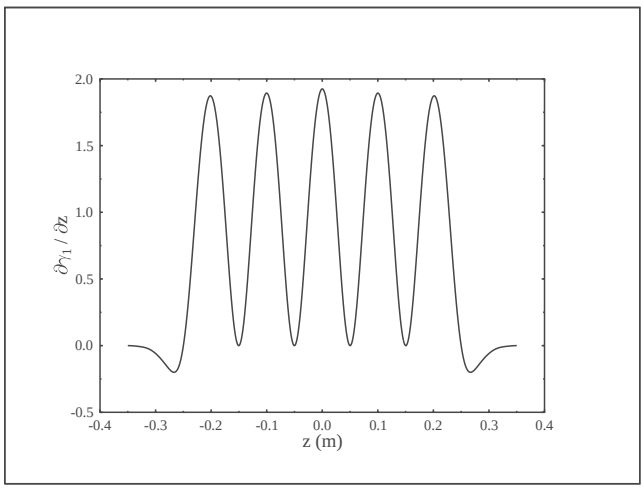

Figure. 2. $\partial\gamma_1/\partial z$ with $\beta = 1$ and $\phi_0 = \pi/2$.

VII. Conclusion

We have shown, in the Hamiltonian context, how to derive symplectic transfer maps in the presence of acceleration. Subsequently we have shown how to derive transfer maps for realistic RF cavities. The method is general purpose and able to handle RF quadrupole and higher multipoles, limited only by the availability of Fourier coefficients of the expansion coefficients.

Acknowledgments

I would like to thank Hongxiu Liu for pointing out reference [7] and for providing the expansion coefficients of the basic RF field, Zenghai Li for pointing out some errors in the multipole expansion and for providing the coefficients for CEBAF cavities from MAFIA data, Filippo Neri and David Douglas for discussions initiating this approach.

References

[1] R. Servranckx et. al., *DIMAD manual*
[2] G. A. Krafft, J. Jackson and D. Douglas, *First Order Focusing and Steering of Misaligned Cavities*, CEBAF TN 90-287.
[3] A. J. Dragt et. al., *Marylie 3.0, A Program for Charged Particle Beam Transport Based on Lie Algebraic Methods*
[4] J. M. Jowett, *Electron Dynamics with Radiation and Nonlinear Wigglers*, CERN Accelerator School 87-03 (1987).
[5] A. J. Dragt, *Numerical third-order transfer map for solenoid*, N.I.M. A298 (1990) 441-459.
[6] A. J. Dragt, F. Neri, J.B.J. van Zeijts and J. Diamond, *Numerical third-order transfer map for combined function dipole*, N.I.M.
[7] K. T. McDonald, *Design of the Laser-Driven RF Electron Gun for the BNL Accelerator Test Facility*, Princeton University preprint, Appendix A (1988).
[8] J.D. Jackson, *Classical Electrodynamics*
[9] Z. Li, *Beam Dynamics in the CEBAF Superconducting Cavities*, Ph.D. Thesis, College of William and Mary, March 1995.
[10] J. van Zeijts, *New Features in the Design Code Tlie*, CAP'93 proceedings and CEBAF-PR-93-003.