OPTIMAL TRANSPORT OF LOW ENERGY PARTICLE BEAMS[*]

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The transport and matching problem for a low energy transport system is approached from a control theoretical viewpoint. We model the beam dynamics and transport section using the KV envelope equations. To this model we apply the principles of optimal control to formulate techniques which aid in the design of the transport and matching section. The techniques are applied to the example of an H- beam transport and matching system.

I. INTRODUCTION

The design of particle beam transport and matching systems has typically been accomplished in much the same way an experiment is run. A computer program is used to simulate the behavior of the beam in a given transport system. The knobs of this simulated system are then adjusted until a satisfactory solution is obtained. This can be a lengthy and arduous process. The progress of such a procedure relies completely upon the experience, judgement, and intuition of the designer.

It is the goal of this work to utilize the principles of optimal control theory to aid in the design of beam transport and matching systems. We have developed an automated technique which determines the optimal lens strengths to match the beam envelope to a prescribed final state. In this paper, we consider beams with elliptical symmetry and apply the results to example Low Energy Beam Transport (LEBT) sections where space charge plays a dominant role.

II. MATHEMATICAL MODEL

A. Beam Dynamics

We model the particle beam using the KV envelope equations. In the two-dimensional steady-state case these equations model a uniform density beam with elliptical cross-section. Let $X(z)$ and $Y(z)$ represent the beam envelope semi-axes in the x and y planes, respectively. This system may be described by the system of coupled differential equations [1]

$$\begin{aligned} X'' + \kappa_x(z)X - \frac{2K}{X+Y} - \frac{\varepsilon_x{}^2}{X^3} = 0, \\ Y'' + \kappa_y(z)Y - \frac{2K}{X+Y} - \frac{\varepsilon_y{}^2}{Y^3} = 0, \end{aligned} \tag{1}$$

where the prime indicates differentiation with respect to z, K is the generalized beam perveance, and ε_x and ε_y are the effective emittances of the beam in the x and y planes, respectively. The functions $\kappa_x(z)$ and $\kappa_y(z)$ represent the action of the transport section in the x and y planes, respectively. They are usually referred to as the focusing or control functions. These equations also describe the behavior of the r.m.s. beam envelope for any beam with elliptical symmetry in the xy plane [1,2].

B. The Transport System

The physical transport section consists of N discrete focusing lenses cascading axially. One of the most important assumptions we make in the paper is that <u>the action of each lens is independent of the others</u>. That is to say that while the beam propagates through a lens, no other lens affects it. This is not always physical, since we know that, in the case of electrostatic or magnetostatic quadrupole lenses, the fields of one lens tend to leak into the regions of any adjacent lenses. Typically, however, we may neglect any small coupling of this type and still aquire accurate results. Consequently, for each lens the focusing function is nonzero only on a finite interval of the z axis.

C. Boundary Conditions (Matching)

We are given initial conditions for the beam envelope at the transport section's entrance position, $z=z_i$. Label these initial conditions (X_j,X_i') for the x plane and (Y_j,Y_i') for the y plane. In the case of a transport and matching system we are also given desired final conditions at $z=z_f$, the exit location of the transport section. We will call these conditions (X_f,X_f') and (Y_f,Y_f') for the x and y planes, respectively. Thus, we must satisfy the following boundary conditions along with Eq. (1):

$$\begin{aligned} X(z_i) &= X_i, & X(z_f) &= X_f, \\ X'(z_i) &= X_i', & X'(z_f) &= X_f', \\ Y(z_i) &= Y_i, & Y(z_f) &= Y_f, \\ Y'(z_i) &= Y_i', & Y'(z_f) &= Y_f', \end{aligned}$$

D. Cost Functional

Let $X(z)$ and $Y(z)$ represent the x and y envelopes of some desirable reference trajectory for the system (these functions would be chosen by the designer). If the pair $[X(z),Y(z)]$ is the actual solution to Eq. (1) for a given system, then a plausible merit functional J for the solution trajectory is given by

$$J\big[X(z),Y(z)\big] \equiv \frac{1}{2}\int_{z_i}^{z_f}\Big[\big(X(z)-\overline{X}(z)\big)^2 + \big(Y(z)-\overline{Y}(z)\big)^2\Big]dz. \tag{3}$$

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[*] Supported by DOE and ONR

This functional, in essence, measures the distance between the solution trajectory [$X(z),Y(z)$] and the reference trajectory [$X(z),Y(z)$].

The boundary conditions of Eq.'s (2) may be included in the functional J with the addition of a boundary term Φ

$$\Phi\left[X\left(z_f\right),Y\left(z_f\right)\right] \equiv \frac{1}{2}\left[X\left(z_f\right)-X_f\right]^2+\frac{1}{2}\left[Y\left(z_f\right)-Y_f\right]^2 . \qquad (4)$$

Thus, the functional which we actually minimize is $J\left[X(z),Y(z)\right]+\Phi\left[X\left(z_f\right),Y\left(z_f\right)\right]$.

III. OPTIMIZATION TECHNIQUE

Since the lens actions are independent, the focusing function $\kappa(z)$ may be subsectioned into N discrete parts, one for each lens. Denote the focusing function for lens n as $\kappa_n(z)$. Also the functional profile for each $\kappa_n(z)$ is known from the geometry of the lens. Therefore, only the amplitude of each $\kappa_n(z)$ remains variable (we do not vary the axial placement of each lens). Denote these amplitudes u_n. We are left with a linear cascade of discrete lenses which act on the beam, in succession, according to Eq. (1). The beam is steered solely by adjusting the set of controls $\{u_n\}$. This situation is referred to in the literature as a multistage control network [3]. The formal control problem is stated as: *find the sequence of controls* $\{u_n\}$ *which steers the system state from* [X_i,Y_i] *to* [X_f,Y_f] *according to the dynamics of Eq. (1) and which minimizes the merit functional of Eq. (3).*

We employ two different techniques from optimal control theory to solve this problem. The first is dynamic programming which has been outlined in a previous paper [4]. The technique works well for axisymmetric systems but usually becomes too CPU intensive for the two dimensional KV equations. Rather, in this situation note that Eq. (1) yields $X(z)$ and $Y(z)$ as implicit functions of the u_n's. Therefore the functional J may also be regarded as a function of the u_n's. The next logical step to this method of representation would be to take the gradient of J with respect to the u_n's. Once we have this gradient, we may use nonlinear programming to search for the minimizing set of lens amplitudes [5]. This approach constitutes the second technique for solving the control problem. Fortuitously, this control problem has a rich mathematical structure which may be exploited for computation of the gradients. It is possible to find them using only numerical integration, rather than differentiation. This yields a more accurate and a more stable search algorithm.

The major advantage of the second approach is that it is substantially faster than dynamic programming. So much so that the algorithm usually converges in a matter of minutes (dynamic programming for the fully two dimensional case typically has run times on the order of a day). The major disadvantage is that the technique searches out only local minima. Consequently, it is necessary to pick a starting point for the algorithm. That is, the designer must choose a set of starting values for the u_n's. Once started, the algorithm will pick out local minima in the vicinity of this starting set. This is quite unlike dynamic programming, which is a global technique not requiring any differentiability conditions.

IV. EXAMPLE

Both the algorithms discussed above have been implemented in a computer-aided design program called **Spot**, which runs on the PC under Microsoft Windows. It is an environment where the designer interacts with the optimizer in order to steer it in the desired direction. In this way the designer may quickly obtain local solutions to the optimal control problem using the nonlinear programming technique. Once found, the result may be checked using dynamic programming.

A. LEBT System

We consider the case of a Low Energy Beam transport (LEBT) section for high-current, high-brightness H^- beam currently under study at the University of Maryland [6]. The system is composed of six electrostatic quadrupole lenses (ESQ's) sandwiched between grounding shunts. We model the action of each lens using the "hard-edge" approximation. A detailed description of the system can be found in reference [6]. We list below the relevent parameters.

I (mA)	V (keV)	ε_x, ε_y (m-rad)	X_i & Y_i (mm)	X_i' & Y_i' (mrad)
30	*35*	*5.56x10^{-5}*	*1.25*	*50*

Table 1: Beam Parameters

Lens No.'s	*Aperature Radius(mm)*	*Length (mm)*	*Spacing (mm)*
1&6	*15.0*	*25.0*	*6.0*
2&5	*22.0*	*59.0*	*6.0*
3&4	*22.0*	*47.0*	*6.0*

Table 2: LEBT Parameters

B. Design Guidelines

Our goal is to match the beam to the final state X_f=Y_f=1.25mm, X_f'=Y_f'=-50mrad. We have the design guidelines that the beam excursions through the ESQs should not exceeded 75% of the aperature radius; this requirement will minimize spherical aberrations. Also, the lens voltage seen at the beam envelope should not exceed 10% of the beam voltage in order to minimize chromatic aberrations. In the following figures the controls $\{u_n\}$ are plotted, rather than the actual ESQ voltages. In our nonrelativistic situation, the conversion formula is given as

$$V_n = u_n V_b a^2 \qquad (5)$$

where V_n is the ESQ voltage, V_b is the beam voltage and a is the ESQ aperature radius.

The reference trajectory we have chosen is a piecewise linear function of z (we let $X(z)=Y(z)=R(z)$).

$$\bar{R}(z) \equiv \begin{cases} 0.00125 + 0.100z & if \quad z\varepsilon[0, 0.062] \\ 0.01335 & if \quad z\varepsilon[0, 0.062, 0.233] \\ 0.01335 - 0.100z & if \quad z\varepsilon[0.233, 0.354] \end{cases} \qquad (6)$$

In this way the reference trajectory levels off to 60% of the aperature radius of lenses 2,3,4, and 5 while remaining well within the requirements of lenses 1 and 6. The reference trajectory is shown in the following figures along with the corresponding solutions.

Figure 1 depicts the solution obtained by the nonlinear programming technique without any constraints imposed upon the lens voltages. The beam is essentially "bounced" off of these two lenses. Clearly this is an unacceptable solution since the beam envelope is comparable to the lens aperature. The situation is remedied by imposing constraints on the lens voltages in the nonlinear programming problem.

It was found that the current ESQ system cannot strictly meet the criteria for minimization of chromatic aberrations. A feasible solution was found when holding the lens voltages seen at the beam envelope to 15% of the beam voltage. This solution is shown in Figure 2. Note that the beam focusing is distributed more evenly across the lenses.

We wish to compare these solutions with that obtained previously without any automation. Figure 3 shows a solution obtained strictly by trial and error with the aim of achieving the same design guidelines. Both show similar characteristics. However, the trial and error solution violates more of the design guidelines. The most notable violations occur at the first lens, where the beam envelope fills 85% of the ESQ aperature and the lens voltage at the envelope is 20% of the beam voltage. This solution also fails to meet the boundary conditions exactly; the convergence is only -40mrad. We also mention that the solution of Figure 3 was found over several hours by an experienced designer while that of Figure 2 was found in less than half an hour.

V. CONCLUSION

The optimization techniques discussed here provide the basis for a useful computer aided design tool. When implemented as above, the designer may interactively guide the optimizer to desirable solutions. These solutions may then be checked against the dynamic programming scheme. This saves a substantial amount of time in the design phase and also allows for the exploration of many design alternatives.

VI. REFERENCES

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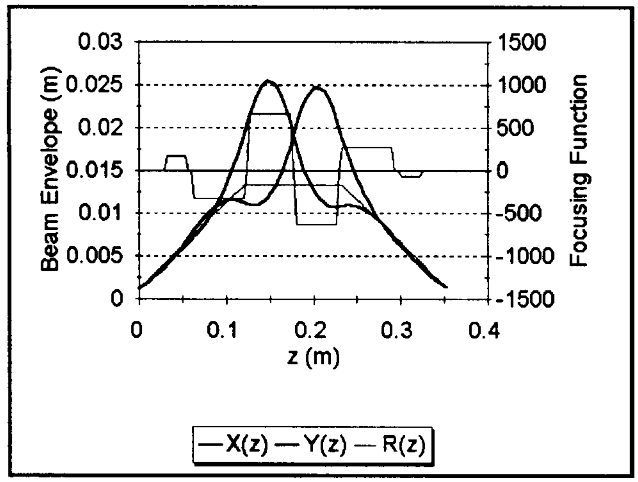

Figure 1: Unconstrained Optimal Solution

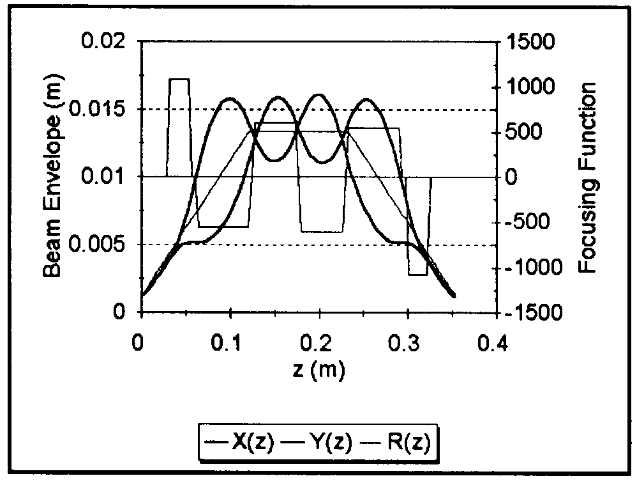

Figure 2: Constrained Optimal Solution

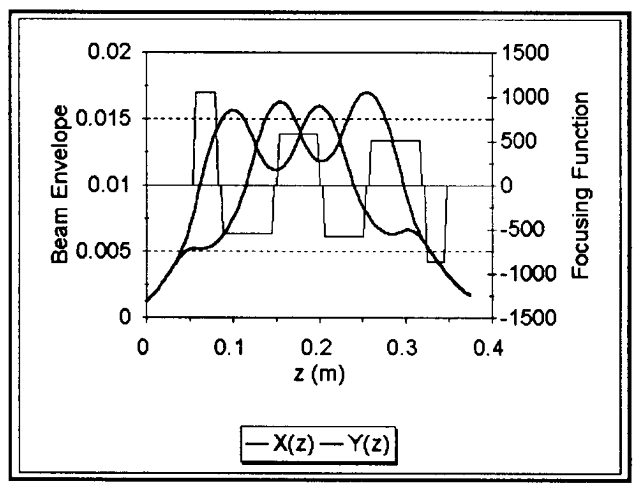

Figure 3: Trial and Error Solution

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