

INVESTIGATION OF THE BEAM IMPEDANCE OF A SLOWLY VARYING WAVEGUIDE

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A perturbation method is used to obtain analytic expressions for the multipole longitudinal and transverse beam impedance for an arbitrary waveguide whose radius is slowly varying and for the specific case of a symmetric small-angle taper. This method is also applicable for a particle in a wiggler undergoing periodic motion.

I. INTRODUCTION AND BASIS OF THE METHOD

In linear colliders, the particle beam traversing the structure will tend to possess a corona of stray particles with large transverse amplitudes. In order to minimize the deleterious effects of these particles on the luminosity of the beam a scraper is often used to disassociate them from the main beam. However, the scraper may lead to an enhanced transverse wake-field and hence lead to a diminishing of the beam emittance.

The method delineated below to calculate the beam impedance, relies on the angle of the taper being small, as is also required in practice to minimize beam degradation. In order that the expansion remain valid it is required that $k_0 b \ll 1$ and $k_0 b b' \ll 1$ (where $b' = db/dz$). The local change in $b(z)$, is required to be small, however the overall change may be large¹.

II. APPLICATION OF METHOD TO THE MONOPOLE LONGITUDINAL IMPEDANCE

In the frequency domain the electric and magnetic field is expressed in terms of \mathbf{A} , the vector potential:

$$\left. \begin{aligned} \mathbf{E} &= -jZ_0(k_0 + k_0^{-1}\nabla\nabla)\mathbf{A} \\ \mathbf{H} &= \nabla\times\mathbf{A} \end{aligned} \right\} \quad (2.1)$$

where k_0 , is the free space wavenumber, Z_0 is the impedance of free space and the vector potential, for monopole a mode, lies along the axis of the structure, $\mathbf{A} = z\mathbf{A}_z$. The wave equation, upon applying the Lorentz condition, for a charge Q traveling with a velocity $v_z (= c/\beta_z)$ offset from the axis by r_0 , becomes:

$$\left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - 2jk_0 \frac{\partial}{\partial z} + \frac{\partial^2}{\partial z^2} \right) y = -\frac{\delta(r-r_0)}{r} \quad (2.2)$$

Here terms of order γ^2 have been neglected, and the enhanced wavenumber and axial potential are given by:

$$\left. \begin{aligned} \hat{k}_0 &= k_0 / \beta_z = k_0 c / v_z \\ 2\pi\mathbf{A}_z &= Qye^{-jk_0 z} \end{aligned} \right\} \quad (2.3)$$

The wave equation, (2.2), is solved iteratively using the perturbation procedure outlined in the previous section. Performing iterations about the zero order equation allows the following equations to be obtained for the first order and n-th order iterations:

$$\left. \begin{aligned} \nabla_r^2 y^0 &\equiv \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} y^0 = -\frac{\delta(r-r_0)}{r} \\ \nabla_r^2 y^n &= 2jk_0 \frac{\partial}{\partial z} y^{n-1} - \epsilon_n^1 \frac{\partial^2}{\partial z^2} y^{n-2} \end{aligned} \right\} \quad (2.4)$$

where $\epsilon_n^1 = 1 - \delta_n^1$ and δ_n^1 is the Kronecker delta function. The Green's function for the left hand side of the zero order part of (2.4), viz, $G(r, r') = r' \ln(r/r')$, allows the general solution to the above equation to be developed as:

$$y^{(n)} = \int_0^r dr' \left\{ 2jk_0 \frac{\partial}{\partial z} y^{n-1} - \epsilon_n^1 \frac{\partial^2}{\partial z^2} y^{n-2} \right\} G(r, r') + a_n(z) \quad (2.5)$$

where $a_n(z)$ is a constant of integration and the quantity in parentheses is evaluated at $r=r'$. Thus the zero order, and first order solution are obtained as:

$$\left. \begin{aligned} y^{(0)} &= a_0(z) - \ln(r/r_0)\theta(r-r_0) \\ y^{(1)} &= a_1(z) + jk_0 r^2 a_0'(z) \end{aligned} \right\} \quad (2.6)$$

where θ is the unit step function. The $a_n(z)$ functions are obtained upon consideration of the boundary condition that the electric field along the taper is zero along the plane of the transition:

$$E_z + b'(z)E_r = 0 \quad (2.7)$$

The above boundary condition is applied successively at each iteration:

$$\frac{d}{dz} y^n + \frac{\partial}{\partial z} y^n = \frac{\epsilon_n^0}{jk_0} \frac{d}{dz} \left[\frac{\partial}{\partial z} y^{n-1} \right] \quad (2.8)$$

In the above, the total derivatives are evaluated taking into account $b(z)$ variation. This allows the wave equation to be solved in powers of k_0 . The longitudinal impedance is given by the inverse Fourier transform of the wake field and this is readily rewritten in terms of the electric field as:

$$Z_L = -\frac{1}{Q} \int_{-\infty}^{+\infty} dz E_z(k_0) \exp(jk_0 z) \quad (2.9)$$

This is transformed into:

$$Z_L = j \frac{Z_0}{Q} \int_{-\infty}^{+\infty} dz \left(k_0 + k_0^{-1} \frac{\partial^2}{\partial z^2} \right) A_z \quad (2.10)$$

and integrating by parts enables the impedance to be obtained as:

$$Z_L = \frac{Z_0}{\pi} \sum_{i=1}^n y^{(i)} \quad (2.11)$$

The impedance resulting from the application of this method up to third order in k_0 is given by:

$$Z_L = \frac{jk_0 Z_0}{4\pi} \int_{-\infty}^{+\infty} dz \left[b'^2 + \frac{5}{24} b'^4 + b^2 b'^2 \left(\frac{1}{8} - \frac{k_0^2 b^2}{12} \right) \right] \quad (2.12)$$

Comparing the above with the impedance obtained by Yokoya² it is evident that the first term in parentheses corresponds to his result and all additional terms are higher order corrections.

Applying this method to the impedance of a symmetric cosinusoidal taper, $b(z) = b_0 - d \cos^2\left(\frac{\pi z}{2g}\right)$ gives³:

$$Z_L = \frac{jk_0 g \alpha^2}{4\pi} \left\{ 1 - \frac{1}{2} \left(\frac{\pi b_0}{2d} \right)^2 (\tilde{z}_1 + \tilde{z}_3) \right\} \quad (2.13)$$

where the three parameters, \tilde{z}_1 , \tilde{z}_3 and α are given by:

$$\left. \begin{aligned} \tilde{z}_1 &\equiv 1 - \frac{d}{b_0} + \frac{3}{4} \left(\frac{d}{b_0} \right)^2, \quad \alpha \equiv \frac{\pi d}{g}, \\ \tilde{z}_3 &\equiv \frac{2k_0^2 b_0^2}{3} \left[1 - \frac{2d}{b_0} + \frac{21}{8} \left(\frac{d}{b_0} \right)^2 + \frac{13}{72} \left(\frac{d}{b_0} \right)^3 + \frac{1052}{1536} \left(\frac{d}{b_0} \right)^4 \right] \end{aligned} \right\} \quad (2.14)$$

Thus, it is evident that for $k_0 b_0 \ll 1$ the first term of (2.14) is sufficient for the calculation of the impedance. However, in the opposite limit higher order terms must be retained.

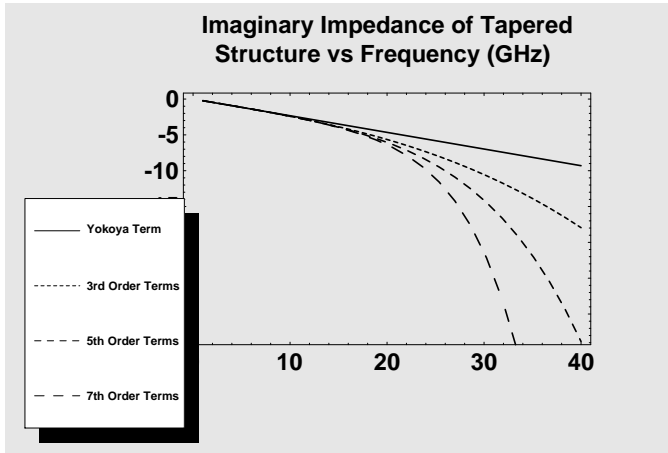


Figure 1

Curves of the impedance function, given by (2.13) up to third order in free space wavenumber, for $b_0 = 1\text{cm}$, $d = .3\text{cm}$, and $g = 6\text{cm}$, are illustrated in figure 1 (where additional terms up to seventh order in k_0 are also included). The linear functional dependence on frequency is indeed sufficient at large wavelengths. However, increasing the frequency rapidly gives rise to significant non-linearity in the dependence of the impedance on k_0 . Indeed, for frequencies in the neighborhood of 30 GHz the perturbation scheme is no longer valid as is revealed upon inspecting higher order perturbations.

III. EVALUATION OF THE TRANSVERSE IMPEDANCE

The transverse impedance is evaluated by solving the wave equation for a vector potential, which in the frequency domain, has components:

$$\mathbf{A} = \sum_{i=-\infty}^{+\infty} e^{jm\phi} \left[A_r \hat{r} + A_\phi \hat{\phi} + A_z \hat{z} \right] \quad (3.1)$$

The wave equation for a harmonic m , is transformed into:

$$\left. \begin{aligned} \left[\nabla_r^2 - \left(\frac{m}{r} \right)^2 \right] y_z &= -\frac{\delta(r-r_0)}{r} + \left[2jk_0 \frac{\partial}{\partial z} - \frac{\partial^2}{\partial z^2} \right] y_z \\ \left[\nabla_r^2 - \left(\frac{m \pm 1}{r} \right)^2 \right] y_\pm &= \left[2jk_0 \frac{\partial}{\partial z} - \frac{\partial^2}{\partial z^2} \right] y_\pm \end{aligned} \right\} \quad (3.2)$$

where the harmonics of the vector potential are given by:

$$\left. \begin{aligned} A_z &= \frac{Q}{2\pi} y_z e^{-jk_0 z} \\ A_r \pm A_\phi &= \frac{Q}{2\pi} y_\pm e^{-jk_0 z} \equiv \frac{Q}{2\pi} (y_r \pm y_\phi) e^{-jk_0 z} \end{aligned} \right\} \quad (3.3)$$

Using the Green's function for the left hand side of (3.2) viz

$$\left. \begin{aligned} G_z(r, r') &= \frac{r'}{2m} \left[\left(\frac{r}{r'} \right)^m - \left(\frac{r'}{r} \right)^m \right] \\ G_\pm(r, r') &= \frac{r'}{|m \pm 1|} \left[\left(\frac{r}{r'} \right)^{|m \pm 1|} - \left(\frac{r'}{r} \right)^{|m \pm 1|} \right] \end{aligned} \right\} \quad (3.4)$$

the wave equation is solved for the m -th order harmonic of the vector potential, expanding about the zero order solution enabling the n -order equation to be obtained in the form:

$$\left. \begin{aligned} y_z^n &= \int_0^r dr' G_z(r, r') f_z + a_z^n(z) (r/r')^{|m|} \\ y_\pm^n &= \int_0^r dr' G_\pm(r, r') f_\pm + a_\pm^n(z) (r/r')^{|m \pm 1|} \end{aligned} \right\} \quad (3.5)$$

where the functions within the integrals are evaluated at $r=r'$ and are given by:

$$\left. \begin{aligned} f_z &= -\delta_n^0 \frac{\delta(r-r_0)}{r} + \epsilon_n^0 \left[2j\hat{k}_0 \frac{\partial}{\partial z} y_z^{n-1} - \epsilon_n^1 \frac{\partial^2}{\partial z^2} y_z^{n-2} \right] \\ f_{\pm} &= \epsilon_n^0 \left[2j\hat{k}_0 \frac{\partial}{\partial z} y_{\pm}^{n-1} - \epsilon_n^1 \frac{\partial^2}{\partial z^2} y_{\pm}^{n-2} \right] \end{aligned} \right\} \quad (3.6)$$

This enables the zero order solution to be obtained as:

$$\left. \begin{aligned} y_z^{(0)} &= a_z^0(z) \left(\frac{r}{r_0} \right)^{|m|} - \theta(r-r_0) \left[\left(\frac{r}{r_0} \right)^{|m|} - \left(\frac{r_0}{r} \right)^{|m|} \right] \\ y_{\pm}^{(0)} &= a_{\pm}^0(z) \left(\frac{r}{r_0} \right)^{|m \pm 1|} \end{aligned} \right\} \quad (3.7)$$

Further, iterations proceed using (3.5) and the remaining constants of integration, $a_z^n(z)$ and $a_{\pm}^n(z)$ are evaluated by applying the condition that both the tangential electric field and the azimuthal electric field are zero along the boundary:

$$\left. \begin{aligned} \hat{D}y &= -\frac{jk_0 b'}{2} (y_+ + y_-) - \frac{d}{dz} (y_z + jk_0^{-1} \hat{D}y) \\ y_z &= -\frac{j}{k_0} \hat{D}y + \frac{jk_0 b}{2m} (y_+ - y_-) \end{aligned} \right\} \quad (3.8)$$

Here \hat{D} operating on y is defined by:

$$\hat{D}y \equiv \left\{ \frac{1}{r} \frac{\partial}{\partial r} r (y_+ + y_-) + \frac{m}{2r} (y_+ - y_-) + \frac{\partial}{\partial z} y_z \right\} \quad (3.9)$$

This completes the calculation of the total field excited by the m -th harmonic of the charge traversing the structure.

The longitudinal impedance is given by:

$$\hat{Z}_L \equiv Z_L(r_0)^m = -\frac{Z_0}{Q} \int_{-\infty}^{+\infty} dz E_z(\omega) e^{jk_0 z} \quad (3.10)$$

and in terms of the vector potential:

$$\begin{aligned} \hat{Z}_L &= \frac{jZ_0}{2\pi} \int_{-\infty}^{+\infty} dz \left[k_0 y_z + e^{jk_0 z} k_0^{-1} \frac{\partial}{\partial z} (\hat{D}y - jk_0 y_z) e^{-jk_0 z} \right] \\ &= -\frac{Z_0}{2\pi} \int_{-\infty}^{+\infty} dz \hat{D}y \end{aligned} \quad (3.11)$$

where (2.1) has been used and an integration by parts has been performed. Further, in cylindrical coordinates the Panofsky-Wenzel theorem⁴ may be applied, enabling the transverse impedance to be obtained as:

$$Z_T = -\frac{m}{k_0} Z_L \quad (3.12)$$

This facilitates the transverse impedance to be obtained up to zero order in k_0 as:

$$Z_T = \frac{jZ_0 m}{\pi(1+m)} \int_{-\infty}^{+\infty} dz \left[\frac{b'}{b^m} \right]^2 \quad (3.13)$$

Here the transverse impedance has been doubled to convert from an exponential variation to a cosinusoidal harmonic. Additional higher order corrections, up to second order in k_0 are readily included for the dipole mode ($m=1$):

$$Z_T = \frac{jZ_0}{\pi} \int_{-\infty}^{+\infty} dz \left\{ \left[\frac{b'}{b} \right]^2 + \frac{1129}{8640} k_0^2 b'^4 \right\} \quad (3.14)$$

IV. DISCUSSION

The perturbation technique is an accurate method to evaluate the impedance of slowly varying accelerator structures consisting of waveguide with a sufficiently slowly varying radius and for a restricted frequency range. For the specific taper under consideration a first order perturbation is augmented with additional higher order terms with increasingly large frequencies up to the point at which the perturbation scheme is no longer valid.

Additional work is in progress on extending the frequency range in which the technique is valid and this is achieved by enhancing the method with a higher order perturbational technique. In this case (2.2) becomes:

$$\left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - 2j\hat{k}_0 \frac{\partial}{\partial z} \right) y = -\frac{\delta(r-r_0)}{r} - \frac{\partial^2}{\partial z^2} y \quad (4.1)$$

The zero order part of (4.1) corresponds to setting the right hand side to zero. Utilizing this method enables the back-scattered wave to be taken into account and this enables the real component of the impedance to be evaluated. Further work is also in progress on applying this technique to investigate the beam impedance of a FEL wiggler.

V. REFERENCES

1. For a harmonic oscillator, driven at a frequency substantially below its natural frequency a perturbation about the zero order (obtained by setting all time derivatives to zero), gives an accurate solution for the amplitude of the oscillation. The method described herein is the spatial analogue of the time-dependent perturbation of a simple harmonic oscillator.
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