PARTICLE TRACKING WITH GENERATING FUNCTIONS OF MAGNETIC FRINGING FIELDS *

Godehard Wüstefeld, BESSY II Project
Rudower Chaussee 5, Geb. 15.1, 12489 Berlin, Germany

Abstract

A construction scheme for generating functions (GFs) suitable for particle tracking across arbitrary magnetic fields is presented. The GF is approximated by a power series solution of the Hamilton-Jacobi differential equation [1] with analytical expressions for the coefficients. This approach is applied to magnetic fringing fields, which are presented as simplified analytical expressions. A short REDUCE [2] code transforms the vector potentials automatically into an expanded GF.

I. Construction of Generating Functions

Generating functions (GFs) are an excellent tool for particle tracking across magnetic fields. They perform a canonical transformation over a finite path length. Even if they are not exactly known and approximate expressions are used, they still preserve the Hamiltonian character of the transformation.

Power series with analytical coefficients as approximations for GFs of given magnetic fields were discussed in [3], with special applications to simulate wigglers and undulators, as used in synchrotron light sources [4]. The present approach is much more direct and transparent. It requires less effort in manipulating power series. A short REDUCE code is sufficient to obtain the GF; using a general method to construct approximated, analytical expressions for GFs, based on the Hamilton-Jacobi equation. Starting with a given vector potential of the magnetic field, the code generates automatically a power series, where the coefficients are analytic functions of the vector potential and the particle coordinates.

The GF is calculated by starting with a Hamiltonian in a fixed cartesian coordinate system, where x, y and z are the horizontal, vertical and longitudinal axes, respectively (for a general discussion see for example [5]):

$$H = -\sqrt{1-(p_x-A_x)^2-(p_y-A_y)^2}-A_z,$$

were p_x and p_y are the transverse canonical momenta, normalized to the full particle momentum and A_x, A_y and A_z are the vector potentials of the fields, normalized to the beam rigidity parameter $B\rho$ which is proportional to the full particle momentum. Expanding the square root of the Hamiltonian yields:

$$H = (p_x-A_x)^2/2+(p_y-A_y)^2/2-A_z.$$

This approximated form of the Hamiltonian used for the present scheme is not a severe restriction, because higher order terms of the expansion could be taken into account. In most cases, the transverse momenta and the vector potentials are small and a Taylor expansion with respect to these variables is justified.

The GFs used in this paper depend on the initial particle momenta p_{xi}, p_{yi} and on the final position variables x, y and z. They are special kinds of GFs of a more general type, that is often abbreviated as F_2. Particle coordinates are derived from partial derivatives of the GF:

$$p_x = \partial F/\partial x,\ p_y = \partial F/\partial y,\ x_i = \partial F/\partial p_{xi},\ y_i = \partial F/\partial p_{yi},$$

were x_i,y_i are the initial coordinates and p_x,p_y are the final particle momenta. In this paper the GF is defined always at the starting point $z = 0$ and at the end point z of the transformation in a fixed cartesian coordinate system. The GF yields implicit expressions for the transverse coordinate relations; using them as a tool for tracking requires normally a Newton-Raphson root-finding method for solving the transformation.

The Hamilton-Jacobi equation is a first order differential equation, which gives a local relation between the Hamiltonian (H) and the GF (F):

$$H + \partial F/\partial z = 0.$$

In this equation the momenta are replaced by derivatives of the GF, $p_x = \partial F/\partial x$, $p_y = \partial F/\partial y$. To solve it with respect to F, a series expansion of F is chosen with respect to p_{xi}, p_{yi} and a variable x_3 which counts the order of the A_x, A_y and A_z terms:

$$F = \sum_{l,m,n} f_{lmn}\, p_{xi}^l\, p_{yi}^m\, x_3^n,$$

were the f_{lmn} are dependent on x, y and z. The expansions with respect to the vector potentials are not written explicitely; this is included in the f_{lmn} terms and counted by the variable x_3. The order of the expansion is the sum $l+m+n$.

The Hamilton-Jacobi equation relates different derivatives of the coefficients f_{lmn} and can be easily solved by a recursive method with an algebraic code such as REDUCE. To demonstrate the method, a REDUCE input code for the GF of a quadrupole is given. The code has a PASCAL like input language and is easily readible:

```
% quadrupole example, g = gradient [m^-2]
% input of vector potential:
az := g*(x**2-y**2)/2 ; ax := 0 ; ay := 0
;
% 4th order generating function:
ord := 4 ;
fgen:= 0 ;
for ic:=1:ord do for ix:=0:ord do
for iy:=0:ord do for i3:=0:ord do
if ix+iy+i3=ic then
<< depend f(ix,iy,i3),x,y,z ;
fgen := fgen +
```

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```
f(ix,iy,i3)*pxi**ix*pyi**iy*x3**i3 >> ;
% initial values of the series:
f(1,0,0) := x ; f(2,0,0) := -z/2 ;
f(0,1,0) := y ; f(0,2,0) := -z/2 ;
% limitation of terms:
for ix:=0:ord+1 do for iy:=0:ord+1 do
for i3:=0:ord+1 do if ix+iy+i3>ord then
let pxi**ix*pyi**iy*x3**i3 = 0 ;
% Hamiltonian:
ham:=(px-x3*ax)**2/2+(py-x3*ay)**2/2-x3*az
;
% substitution of the momenta:
ham:=sub(px=df(fgen,x),py=df(fgen,y),ham)
;
% Hamilton-Jacobi equation:
hamjac := ham + df(fgen,z) ;
% iterative solution of the equation:
for ic:=1:ord do for ix:=0:ord do
for iy:=0:ord do for i3:=0:ord do
if ix+iy+i3=ic then
<< term := coeffn(coeffn(coeffn(hamjac,
pxi,ix),pyi,iy),x3,i3) ;
if term neq 0 then <<
sol1 := solve(term=0,df(f(ix,iy,i3),z)) ;
sol2 := rhs(part(sol1,1)) ;
sol3 := int(sol2,z) ;
f(ix,iy,i3) := sol3-sub(z=0,sol3) >> >> ;
% printing of the generating function:
sub(x3=1,fgen) ;
;end;
```

The code prints the 4th order power series for the GF as:

$$\begin{aligned} F \;=\;& x^2\, g\, z\,(1/2 - g\,z^2/6 + g^2\,z^4/15 - 17\,g^3\,z^6/630) \\ & - y^2\, g\, z\,(1/2 + g\,z^2/6 + g^2\,z^4/15 + 17\,g^3\,z^6/630) \\ & + p_{xi}\, x\,(1 - g\,z^2/2 + 5\,g^2\,z^4/24 - 61\,g^3\,z^6/720) \\ & + p_{yi}\, y\,(1 + g\,z^2/2 + 5\,g^2\,z^4/24 + 61\,g^3\,z^6/720) \\ & + p_{xi}^2\, z\,(-1/2 + g\,z^2/6 - g^2\,z^4/15) \\ & + p_{yi}^2\, z\,(-1/2 - g\,z^2/6 - g^2\,z^4/15). \end{aligned}$$

From this GF the linear transfer matrix can be calculated, which agrees at least to the 3rd order in g with the well known transfer matrix of a quadrupole. For example, the matrix element m_{11} derived from the GF yields $m_{11} = 1 + gz^2/2 + g^2z^4/24 + g^3z^6/720 + 11g^4z^8/320$. The expanded form of the correct solution differs in the g^4 term: $m_{11} = \cosh(\sqrt{g}z) = 1 + gz^2/2 + g^2z^4/24 + g^3z^6/720 + g^4z^8/40320\ldots$

This simple code can calculate the GFs for most two dimensional normal and skew magnetic fields. One only needs to replace `az:=...` in the first line by the appropriate longitudinal vector potential. Wigglers and undulators can be simulated by the three dimensional field approximation:

```
ax:=cos(kx*x)*cosh(ky*y)*sin(k*z)*b0/k ,
ay:=sin(kx*x)*sinh(ky*y)*sin(k*z)*b0*
kx/(k*ky)
```

and `az:=0`. However, this last example requires a lot of REDUCE working space; because many terms are generated, a 3rd order run is recommended. [1] For more complicated vector potentials one has to modify the code. A practical way is to solve the Hamilton-Jacobi equation for a general vector potential and insert it afterwards into the solution of the explicit vector potential [7].

As an example the second order result of the general expansion of the GF is presented here:

$$\begin{aligned} f_{001} &= \int A_z\, dz \\ f_{002} &= -\int (\,(A_x - \int \partial A_z/\partial x\, dz)^2 \\ &\qquad + (A_y - \int \partial A_z/\partial y\, dz)^2\,)\, dz/2 \\ f_{101} &= \int (A_x - \int \partial A_z/\partial x\, dz\,)\, dz \\ f_{011} &= \int (A_y - \int \partial A_z/\partial y\, dz\,)\, dz \\ f_{100} &= x \qquad f_{010} = y \\ f_{200} &= -z/2 \qquad f_{020} = -z/2 \end{aligned}$$

where the integration ranges from 0 to z. In a similar form the 3rd and 4th order expansions can be constructed [7].

Using the REDUCE code, the GF can be further manipulated to find a form appropriate for a Newton-Raphson fit routine.

As a special application of this method, GFs for magnetic fringing fields can be calculated.

II. Description of Fringing Fields

Magnets such as dipoles, quadrupoles, sextupoles and so on are described by two dimensional multipoles. This two dimensional approximation fails at the ends of these magnets, where the particle beam enters or exits the magnet, and the field strength approaches zero. Three dimensional fields are necessary to describe their longitudinal dependencies. A simple analytical description of the vector potentials of fringing fields is necessary for manipulations with the REDUCE code when constructing the GF. The analytical description discussed here is a simplification and takes into account only the first leading term of the three dimensional field.

Expressions for the fringing fields are derived by starting with the magnetic scalar potential in an expanded form in cylindrical coordinates as:

$$V = \sum_{i=0}^{\infty} a_i(z) r^i \sin m\psi,$$

were r is the radial coordinate with $x = r\cos\psi$, $y = r\sin\psi$. The number m describes the rotational symmetry around the longitudinal z-axis, and ψ is azimuthal angle with respect to the z axis. Quadrupole symmetry is obtained for $m = 2$. If a is independent of z, the two dimensional multipoles are generated. Replacing the $\sin m\psi$ by a $\cos m\psi$ function yields the skew field terms. From the Maxwell condition $\Delta V \equiv 0$ follows (with $a_i' = \partial a_i/\partial z$):

[1] this requires 1 minute cpu time on the DEC 3000 machine (alpha)

$$\sum_{i=0}^{\infty} a_i (i^2 - m^2) r^i + a_i'' r^{i+2} \equiv 0,$$

which yields for the coeffecients the condition: $a_0 = 0$, $a_1(1-m^2) = 0$ and $a_i(i^2 - m^2) + a_{i-2}'' = 0$. This gives a construction law for the series; if the z-dependence of one coefficient a_i is known the higher order terms can be calculated. The coefficients a_0 and a_1 are zero, if the dipole case (m=1) is excluded.

A 3rd order function will be used to describe the z-dependence of a_m. This will limit the series to the first term of the higher order three dimensional multipole. For the dipole, the leading term of higher order is y^3 dependent, and for quadrupoles it is a r^4 term. At least a 3rd order function is required to transform the constant, z-independent field inside of the multipole by a smooth, analytic function to zero, at the end of the multipole:

$$a_m = a_{m0}(u^3 - 3u + 2)/4,$$

where $-1 < u = (z - z_0)/z_0 < +1$, and z_0 is an adjustable, characteristic length of the fringing field extension, typically the magnet aperture radius.

The factor a_{m0} has to be adjusted to the strength of the magnetic field. At the positions $z = 0$, $z = z_0$ and $z = 2z_0$ the function a_m yields a_{m0}, $a_{m0}/2$ and 0. Outside of this interval a_m is fixed to a_{m0} for $z < 0$ and to 0 for $z > 2z_0$. This derivation is valid for one side of the multipole, the fringing field of the other side has to be constructed in a similar way.

For the scalar potential a superposition of the two and three dimensional field terms is given as:

$$V = a_{m0} r^m \sin m\psi \{\, u^3 - 3u + 2 - 3ur^2/(2z_0^2(m+1)) \,\}.$$

and for the dipole:

$$V = a_{10}\, y\, (\, u^3 - 3u + 2 - uy^2/z_0^2\,)/4.$$

The scalar potential of the pseudo-multipole is proportional to a_m''. The integrated value of a_m'' over the full fringing field area is zero, because a_m' vanishes at the boundaries of the fringing field. Integrating only over half of the area, yields a nonvanishing contribution:

$$-\int_0^{z_0} a_m''\, dz = \int_{z_0}^{2z_0} a_m''\, dz = a_m'\big|_{z_0}^{2z_0} = a_m'(z = z_0).$$

A particle that crosses this area will experience two successive kicks around $z = 0$, in opposite directions. Particles circulating many turns in a storage ring, will experience an accumulated effect, if, on the average, their trajectory is inclined with respect to the z-axis, which is the case if the Twiss parameter α differs from zero. Also chromatic effects could be seen if the dispersion function in the fringing field differs from zero.

For the construction of the GF the vector potential is required; it is not uniquely defined. Choosing $A_z = 0$ one obtains from the scalar potential in carteesian coordinates:

$$A_x = -\int \partial V/\partial y\, dz\,, \qquad A_y = \int \partial V/\partial x\, dz.$$

These vector fields can be used for the REDUCE input code. If the number of terms becomes too large, one has to use a modified version of the presented code. In case of quadrupole end fields, for example, one expects influences from fourth order terms. A fourth order expansion of the GF with respect to x, y, p_{xi} and p_{yi} should be sufficent to study typical nonlinear effects of these fringe fields.

III. Applications

Fringing field terms of quadrupoles were included into a standard optics code, based on the analytical representation of the GF. The routines were modulated as invisible insertions for the linear optics, following the scheme proposed by [6]. The linear transformation becomes exactly invisible, if the inverse linear transformation is derived from the GF. The BESSY II optics [8] was checked using these routines. The dynamic aperture and the chromatic behavior show only minor changes, when these routines are activated. From the dipoles no further effect on the optics are expected because the vertical Twiss α at their locations is too small.

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