Stochastic Effects in Real and Simulated Ion Beams

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Abstract

Emittance growth due to intra-beam scattering effects in charged particle beams is estimated using a second order moment description based on the Vlasov-Fokker-Planck equation. It is furthermore shown that numerical emittance growth phenomena in computer simulations of particle beams can also be described by this approach.

1 INTRODUCTION

If we want to analyze emittance growth phenomena in charged particle beams, we must distinguish the following sources:

- a non-self-consistent charge density distribution of the beam, which leads to a redistribution of the beam charges within a few plasma periods. In case that a more homogeneous charge density evolves, this results in a rapid increase of the rms-emittance.
- a temperature imbalance between different degrees of freedom within the beam. This can also be interpreted as a transition from a non-stationary to a stationary solution of the Vlasov equation, usually designated as the equipartitioning effect. The related time constants are considerably smaller, hence the time the beam needs to reach a temperature balance is much larger than the time needed for the charge readjustment.
- various orders of resonances between the beam and the external focusing devices. Since usually only a fraction of the beam particles is in resonance, these effects lead to a degradation of the beam emittance due to halo formation.

All these effects have in common that they can be calculated – at least in principle – by integrating the Vlasov equation, which is a strictly deterministic formulation. In other words, all these phenomena could also be observed if the space charge fields were smooth functions of the spatial coordinates, i.e. in a rigid continuous description.

However, there are effects which originate in the fact that the charge distribution is granular on a microscopic scale. The individual particle motion is thus no longer only governed by the smooth macroscopic space charge field, but also by a rapidly fluctuating field caused by neighboring beam particles. Obviously, these effects can only be tackled by means of statistical methods.

We restrict ourselves to cases where these effects can be classified as a chain of `Markov' processes. Considering beam dynamics, this is always a good approximation. The dynamics of these processes can then be described by the Fokker-Planck equation. Additionally taking into account stochastic effects means that the Vlasov equation has to be replaced by the combined Vlasov-Fokker-Planck equation. It forms the starting point for any physical process, where the macroscopic (smooth) aspect is described by the Liouville equation and where in addition a chain of Markov processes cannot be neglected.

2 FOKKER-PLANCK APPROACH

We start our analysis writing down formally the generalized Liouville theorem:

$$\frac{df}{dt} = \left[\frac{\partial f}{\partial t}\right]_{\mathrm{FP}} \tag{1}$$

Herein $f = f(\vec{x}, \vec{p}; t)$ denotes the normalized 6-dimensional μ-phase space density function that represents a charged particle beam. The l.h.s. of (1) can be expressed in terms of the Vlasov equation

$$\frac{\partial f}{\partial t} + \vec{p} \cdot \vec{\nabla}_x f + \frac{1}{m}\left(\vec{F}^{\mathrm{ext}} + q\vec{E}^{\mathrm{sc}}\right) \cdot \vec{\nabla}_p f = \left[\frac{\partial f}{\partial t}\right]_{\mathrm{FP}} \tag{2}$$

The r.h.s. of (1) is supposed to describe additional stochastic effects not covered by the Vlasov approach. If these effects constitute a Markov process, we can describe it with the help of the Fokker-Planck equation:

$$\left[\frac{\partial f}{\partial t}\right]_{\mathrm{FP}} = \sum_i \frac{\partial}{\partial p_i}\left\{\beta_{f;i} \cdot p_i f\right\} + \sum_{i,j} \frac{m^2 \partial^2}{\partial p_i \partial p_j}\left\{D_{ij}(\vec{p}, t) f\right\} \tag{3}$$

The Fokker-Planck coefficients, namely the diffusion tensor elements D_{ij} and the drift vector components F_i must be determined in an appropriate way depending on the nature of the stochastic process.

3 SECOND ORDER MOMENT EQUATIONS

Applying Sacherer's formalism[1] to the Vlasov-Fokker-Planck equation (2), we get the following coupled set of second order moment equations[5]:

$$\frac{d}{ds}\langle x^2\rangle - 2\langle xx'\rangle = 0 \tag{4}$$

$$\frac{d}{ds}\langle xx'\rangle - \langle x'^2\rangle + k_x^2(s)\langle x^2\rangle - \frac{q\langle xE_x\rangle}{mc^2\beta^2\gamma^3} + \frac{\beta_{f;x}}{c\beta\gamma}\langle xx'\rangle = 0$$

$$\frac{d}{ds}\langle x'^2\rangle + 2k_x^2\langle xx'\rangle - \frac{2q\langle x'E_x\rangle}{mc^2\beta^2\gamma^3} + \frac{2\beta_{f;x}}{c\beta\gamma}\langle x'^2\rangle - \frac{2\langle D_{xx}\rangle}{c^3\beta^3\gamma^3} = 0$$

The similar sets of equations hold for the y- and z-directions.

A useful measure for the beam quality is given by the rms-emittance $\varepsilon_x(s)$, defined as

$$\varepsilon_x^2(s) \;=\; \langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2 \;\; . \tag{5}$$

On the basis of Eqs. (4), the derivative of the rms-emittance (5) is readily calculated to give

$$\frac{1}{\langle x^2\rangle}\frac{d}{ds}\varepsilon_x^2(s) = -2\left(\frac{\beta_{f;x}}{c\beta\gamma}\frac{\varepsilon_x^2(s)}{\langle x^2\rangle} - \frac{\langle D_{xx}\rangle}{c^3\beta^3\gamma^3}\right) \;, \tag{6}$$

neglecting the "excess field energy" terms. For intrinsically matched beams, this quantity is approximately a constant of motion. The remaining terms are related to the Fokker-Planck coefficients to be discussed now.

4 GROWTH RATES

If the diffusion as well as the friction effects can be approximately treated as isotropic, then only one diffusion coefficient D in conjunction with a single friction coefficient β_f appears in our equations:

$$D \equiv \langle D_{xx}\rangle = \langle D_{yy}\rangle = \langle D_{zz}\rangle \quad , \quad \beta_f \equiv \beta_{f;x} = \beta_{f;y} = \beta_{f;z}$$

Under these circumstances, D turns out to be proportional to the "dynamical friction coefficient" β_f [3]:

$$D = \beta_f \cdot \frac{kT}{m} \tag{7}$$

If we define the ratio r_{xy} of the y- to the x-"temperature" as

$$r_{xy}(s) = \frac{T_y}{T_x} = \frac{\varepsilon_y^2(s)}{\langle y^2\rangle}\frac{\langle x^2\rangle}{\varepsilon_x^2(s)} \;, \tag{8}$$

Eq. (6) can be written in an alternative form:

$$\frac{d}{ds}\ln\varepsilon_x^2(s) = \frac{2k_f}{3}\;\left(r_{xy}(s) + r_{xz}(s) - 2\right) \;, \tag{9}$$

wherein $k_f = \beta_f/c\beta\gamma$. Adding Eq. (9) to the corresponding equations for $\ln\varepsilon_y^2$ and $\ln\langle\delta^2\rangle$, after integration we get the following simple expression for the e-folding time τ_{ef} of the total phase space volume

$$\tau_{\mathrm{ef}}^{-1} = \tfrac{1}{9}\,\beta_f\,\left(I_{xy} + I_{xz} + I_{yz}\right) \;, \tag{10}$$

with I_{xy}, I_{xz}, and I_{yz} denoting the three possible integrals of the temperature ratio functions. For example, the dimensionless quantity I_{xy} is given by:

$$I_{xy} = \frac{1}{S}\int_0^S \frac{\left[1 - r_{xy}(s)\right]^2}{r_{xy}(s)}\,ds \geq 0 \;. \tag{11}$$

If the transverse particle dynamics can be decoupled from the longitudinal one, the e-folding time for the transverse emittance evaluates to:

$$\tau_{\perp,\mathrm{ef}}^{-1} = \tfrac{1}{4}\,\beta_f\;I_{xy} \;\; . \tag{12}$$

This condition is fulfilled if – for example – we simulate the transformation of an unbunched charged particle beam using a x,y-Poisson solver.

5 INTRA-BEAM SCATTERING EFFECTS IN STORAGE RINGS

The elementary events for the global process of emittance growth due to intra-beam scattering are Coulomb collisions of individual beam particles. The dynamics of these collisions thus forms the starting point to determine the Fokker-Planck coefficients contained in Eq. (3) for this process. Integrating the coupled set of equations (4) will enable us to study the evolution of the beam properties including intra-beam scattering effects. Explicitly, the coefficients are determined by averaging the velocity change of a test particle over all impact parameters, and subsequently by averaging over all particle velocities assuming that the velocity distribution is Maxwellian.

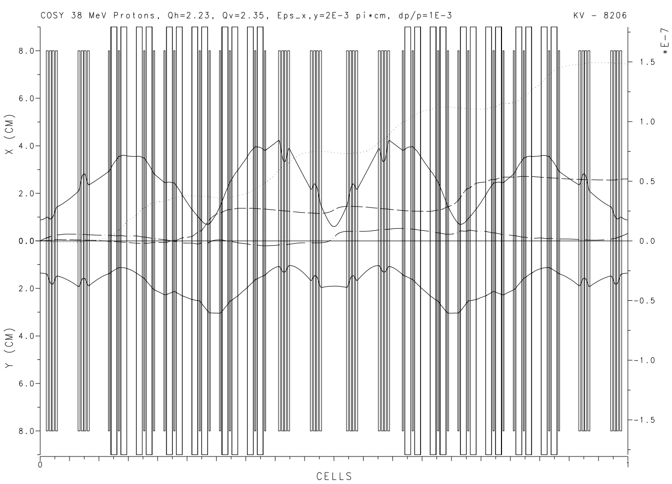

Figure 1: Envelopes and emittance growth functions of a matched beam passing through the Cooler Synchrotron (COSY) at KFA-Jülich. (The scale on the right-hand side applies to the dimensionless emittance growth functions.)

As the result of the averaging procedures, β_f is given by[2, 4]:

$$\beta_f \;=\; \frac{16\sqrt{\pi}}{3}\;n\,c\left(\frac{q^2}{4\pi\epsilon_0 mc^2}\right)^2\cdot\left(\frac{mc^2}{2kT}\right)^{3/2}\;\cdot\;\ln\Lambda\;. \tag{13}$$

In order to gain a better physical insight, this quantity can be expressed alternatively as

$$\beta_f \;=\; \sqrt{\frac{2}{\pi}}\cdot\left(\Delta t_{\mathrm{scattering}}\right)^{-1}\cdot\Gamma^2\cdot\ln\Lambda \quad ,$$

with $\Delta t_{\mathrm{scattering}}$ denoting the average time between two successive scattering events of a beam particle, and Γ the coupling constant of the beam plasma.

As an example, we present an integration of the coupled set (4) based on the structure data of the Jülich Cooler Synchrotron (COSY). The beam parameters used in that calculation are listed in Tab. 1. The results, namely the envelopes and the emittance growth functions along one turn, are plotted in Fig. 1. We observe that the maximum growth of the rms-emittance occurs in the x-direction, whereas for these initial conditions practically no growth occurs in the longitudinal direction.

Table 1: List of Parameters for the COSY intra-beam scattering estimation

ion species	H^{1+}
energy	38 MeV/amu
period length S	183.47 m
horizontal tune Q_h	2.23
vertical tune Q_v	2.35
$\eta_{\rm av} = \gamma^{-2} - \gamma_t^{-2}$	0.371
average beam current I	55 mA
initial RMS emittances $\varepsilon_{x,y}(0)$	5×10^{-6} m
initial RMS momentum spread $\Delta p/p$	5.8×10^{-4}
ellipticity I_{xy}	3.252
ellipticity I_{xz}	2.286
ellipticity I_{yz}	0.997
friction coefficient β_f	4.1935 s^{-2}
horiz. emittance e-folding time $\tau_{x,\rm ef}$	15.0 s
vert. emittance e-folding time $\tau_{y,\rm ef}$	43.0 s
momentum spread e-folding time $\tau_{z,\rm ef}$	336.1 s
total emittance e-folding time $\tau_{\rm ef}$	32.8 s

6 BEAM TRANSPORT SIMULATIONS

As has been mentioned in the introduction, the Fokker-Planck description of stochastic phenomena in the physics of charged particle beams is not restricted to the effect of intra-beam scattering, but applies to any beam dynamical Markov process. As a consequence, the coupled set of moment equations (4) can also be used to explain effects due to random errors generated by the necessarily limited accuracy of computer simulations of particle beams. Especially the problem of calculating the self-fields of arbitrary charge distributions escapes an analytical treatment. The major sources of random errors common to all simulation codes come from the fact that

- the continuously varying self-fields must be replaced by stepwise constant ones,
- the number of simulation particles is much smaller than the real beam particle number.

The joint effect of all these simplifications necessary to keep the computing time finite can be visualized in the way that an additional `error field' of a certain amplitude is added to the `true field' of the real beam. This causes a specific `simulation friction coefficient' $\beta_{f,\rm sim}$ to emerge. Unfortunately, it seems to be impossible to derive a general formula relating globally $\beta_{f,\rm sim}$ to the various sources of `numerical noise'.

In Fig. 2, the emittance growth factors obtained by numerical simulations of a quadrupole channel are plotted for different numbers of space charge calculations (SSC) per cell and different numbers P of simulation particles. It can be observed that the calculated growth rates do not depend very much on the number of space charge calculations. This indicates that in this case the number of 50 space charge calculations suffices to approximate the continuously varying self-fields. On the other hand, the growth rate is halfed if we double the number of simulation particles from 5000 to 10000, i.e.

$$\beta_{f,\rm sim} \propto P^{-1} \quad .$$

This shows that in our case the concept of representative particles constitutes the major source of numerical noise.

For a comparison, the lower curve in Fig. 2 displays the calculated growth rates for a matched beam transformed under the same conditions through a periodic solenoid channel. Consequently, $\beta_{f,\rm sim}$ must have the same value as for the corresponding quadrupole channel transformation. Nevertheless, no rms-emittance growth at all is observed in this simulation. This outcome is explained by Eq. (12), which states that even a positive β_f does not lead to an increase of the rms-emittance if the ellipticity integral (11) vanishes. We conclude that numerical noise phenomena occurring in computer simulations of charged particle beams can adequately be described by the Fokker-Planck approach.

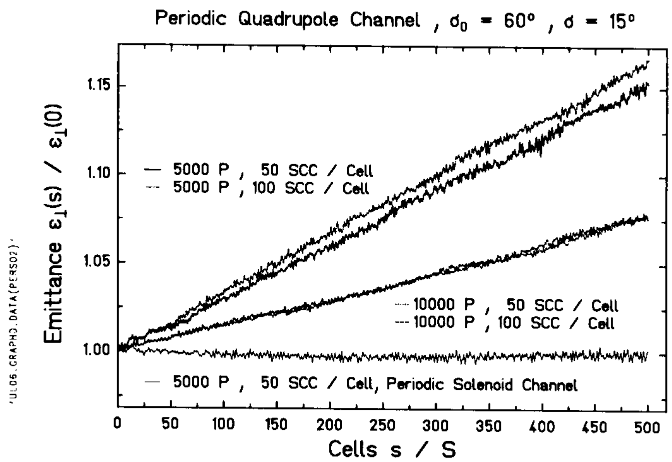

Figure 2: Emittance growth functions obtained by different numerical simulations of a matched beam passing through the GSI experimental quadrupole channel.

7 CONCLUSIONS

The moment description of a charged particle beam has been demonstrated to be useful even if additional stochastic effects must be taken into account. It has been shown that this approach leads to a fairly simple formula which can be used to estimate the growth rates of the beam emittances caused by intra-beam scattering effects. Moreover, since the Fokker-Planck equation describes any Markov process, certain effects of rms-emittance growth in computer simulations can also be explained by this approach.

8 REFERENCES

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