Non-linear Chromaticity Correction with Sextupole Families

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Abstract

The correction of the non-linear chromaticity with sextupoles families is explained by means of a simple perturbation theory. The advantages and limitations of such systems are shown, as well as the constraints they put on machine optics.

I. Introduction

The non-linear chromaticity of the LEP machine at CERN has been successfully corrected with sextupole families for a large variety of different lattices. It works actually so well that the subtleties in this correction have been completely forgotten.

As this type of correction was proposed a long time ago, it is presently felt that more modern systems are better. At the Washington conference in 1993, it was stated that non-linear chromaticity correction of the B-factory PEP2 project with sextupole families was not possible [1]. In fact such a statement might mean that the machine lattice was simply not suitable for such a correction or that the sextupole families were wrongly chosen.

In what follows, the computation of the second derivative of the tune with respect to momentum is recalled first. Then the contribution of periodic chromatic perturbations are estimated. Finally practical applications on how to use sextupole families as well as tolerances concerning the optics to make this use possible are given.

II. Second derivative of the tune with respect to momentum.

When low-β insertions sit at places where the dispersion function is zero, a local correction of their chromaticity is not possible. Consequently an off-momentum mismatch of the insertions appears. For certain tune values, this makes a very large second order derivative of the tune with respect to momentum [2].

A. General expression of the second order tune derivative.

It is relatively straightforward to compute the chromatic dependence of the linear optics parameters by computing the one-turn 2×2 transfer matrix of a machine perturbed by chromatic effects. The calculation is based on the change of the transforms of the β-function due to gradient perturbations, which make it possible to express easily the one-turn transfer matrix. This has been shown in a previous accelerator conference [2]. We call μ the phase advance for one super-period in the machine $\mu = 2\pi$Q if Q is the super-period tune. The ' indicates the derivative with respect to the relative momentum deviation. The second derivative of μ with respect to momentum deviation δ is given by :

$$\mu'' = \mu^{t\,''} + \left[\frac{\alpha\beta'}{\beta} - \alpha'\right]' - \frac{1}{4}\cot\mu\left[\left(\frac{\beta'}{\beta}\right)^2 + \left(\frac{\alpha\beta' - \alpha'\beta}{\beta}\right)^2\right] \quad (1)$$

(Note that in reference [2], there was a miss-print in the μ'' formula : there was one term in excess. The correct formula is the above one). In this formula, the terms on the first line come from the first order tune-shift formula applied to the second order chromatic perturbation per element, i.e. their value is of the same order as the natural μ'. α and β are the standard TWISS parameters [3] taken at an arbitrary origin in the lattice. It is assumed that we are able to compute the transforms of these parameters taking into account chromatic perturbations.

For the computation of the term containing squares, which is the important one, what is needed is the first derivative of the transform of the β-function with respect to δ. This is an important point which had been suspected a long time ago [4], [5] but only formalized only recently [2]. This first derivative is obtained from the derivatives with respect to δ of the integrated gradients at the points of index i, which are $\frac{\partial k_i l_i}{\partial\delta}$. We obtain [4] :

$$\frac{1}{\beta}\frac{\partial\beta}{\partial\delta} = \frac{\beta'}{\beta} = \sum_i \frac{\partial k_i l_i}{\partial\delta}\beta_i \sin 2\left(\mu - \mu_i\right)$$

Taking the derivative of this expression with respect to the longitudinal coordinate, we obtain :

$$\frac{\alpha\beta' - \beta\alpha'}{\beta} = \sum_i \frac{\partial k_i l_i}{\partial\delta}\beta_i \cos 2\left(\mu - \mu_i\right)$$

These two expressions are exactly what is needed to compute the important terms in formula (1). It is essential to recall that β' and α' are not the derivatives of the optics functions with respect to δ, but they are related (not needed here).

In reference [2], the emphasis was put on the contributions of the low-β quadrupoles which make Q" large. It was simply mentioned that the contributions of periodic cells was negligible. We examine it now.

B. Contribution of periodic chromatic perturbations to the second order tune derivative.

To obtain these contributions, we merely compute the sums in equation (1) for $\frac{\partial k_i l_i}{\partial\delta}$ constant, i.e. not depending on the index i. We obtain readily, keeping only the important term, i.e. that one with cotμ :

$$\mu'' \approx -\frac{1}{4}\cot\mu\,\left[\frac{\partial kl}{\partial\delta}\beta\right]^2 \times \left[\left(\sum_i \sin 2\left(\mu - \mu_i\right)\right)^2 + \left(\sum_i \cos 2\left(\mu - \mu_i\right)\right)^2\right] \quad (2)$$

The periodicity of the chromatic perturbation appears in its phase μ_i which is given by :

$$\mu_i = \mu_0 + (i-1)\mu_c$$

μ_0 being the phase of the first perturbation and μ_c the phase between two successive perturbations. The sum of trigonometric functions can be done easily, we obtain finally for n periodic perturbations :

$$\mu'' \approx -\frac{1}{4}\cot\mu\left[\frac{\sin n\mu_c}{\sin\mu_c}\frac{\partial kl}{\partial\delta}\beta\right]^2 \tag{3}$$

Such contributions to μ'' produced for instance by the arc quadrupoles and sextupoles are very small compared with that of the low-β quadrupoles which is [2] :

$$\mu'' \approx -(Kl\beta)^2\cot\mu \tag{4}$$

as the term $\left[\frac{\partial kl}{\partial\delta}\beta\right]^2$ associated with the quadrupoles or the sextupoles of the regular cells is usually smaller than $(Kl\beta)^2$ by two order of magnitude and $\sin n\mu_c$ is smaller than one. Obviously this only is true as long as $\sin\mu_c$ is non zero. This is the case when the chromaticity is corrected with one sextupole family per plane in periodic cells, provided the cell phase advance is different from π.

III. Making sextupole families.

From the preceding argument, we see that if the periodicity of the gradient perturbation is a multiple of π, the fraction $\frac{\sin n\mu_c}{\sin\mu_c}$ is equal to n, so that their contribution is multiplied by n^2. If n is of the order of 10, we see that two order of magnitude can be gained. This can be achieved by forcing the sextupole periodicity to be an odd multiple of π. To make this possible, μ_c must be an odd multiple of $\frac{\pi}{k}$, where k is a small integer different from 1. Then it is possible to assign the same strength to sextupoles separated by k cells, i.e. to build up k sextupole families and to force the sextupole periodicity to be π by assigning different strengths to the families. Under those conditions, sextupole families are an efficient way of making large higher order tune derivatives, especially in large machines thanks to the factor n^2.

For the particular case where it is possible to distribute the sextupoles in families with equal numbers of members, their contribution to μ'', forgetting the quadrupole contributions, is given by :

$$\mu'' \approx -\frac{1}{4}\cot\mu\;\left[\frac{sinnk\mu_c}{\sin k\mu_c}\right]^2\times$$
$$\left[\left(\sum_j \chi_j\sin 2\left(\mu-\mu_{0j}\right)\right)^2+\left(\sum_j \chi_j\cos 2\left(\mu-\mu_{0j}\right)\right)^2\right]$$
$$\text{with : }\chi_i = k'_j l_j D_j\beta_i \tag{5}$$

j is the index of the sextupole families which contain n sextupoles each, D_j is the value of the dispersion function at the sextupole locations and μ_{0j} is the phase of the first sextupole of each family. For $k\mu_c$ multiple of π, we find the factor n^2 in front of the formula. For χ_i independent of i, the sums of trigometric functions are zero.

This formula gives a good idea on the mechanism of second order chromaticity correction with sextupole families. For a practical correction of the non-linear chromaticity, it is necessary to go to a higher order expansion. In fact other perturbation formalisms have been developed for a long time, as [6]. Nevertheless the above formulae tells us that it is important to compensate the first order derivatives of the β-functions when such a correction of the higher order tune derivatives is computed.

IV. Tolerance on the phase advance per cell for periodic sextupole families.

A. General conditions

If the phase advance per cell is not an odd multiple of $\frac{\pi}{k}$ where k is any integer, the factor n^2 disappears. This is what happens for instance if the phase of the regular cells are used to adjust the tunes. The sextupole families have been constructed for a certain value of the phase advance of the regular and this phase advance per cell is subsequently "slightly changed". As a consequence, $nk\mu_c$ may become close to a multiple of π, $k\mu_c$ being not a multiple of π, and the sextupole families loose completely their efficiency as their important contribution to the non-linear chromaticity becomes close to zero.

B. The LEP example

A first good example of non working periodic sextupole families is that of the first LEP lattice [7]. In a superperiod of this machine there was one arc with 30 FODO cells with dipoles and one low-β insertion. The phase advance of the arc cells was "about $60°$ ". It was in fact exactly $60°$ in the horizontal plane but it was close to $55°$ in the vertical plane. The number of cells between two successive sextupoles in a given family was set to 3 because of the "about $60°$ " per cell. This makes 10 sextupoles per family. Then, for the vertical plane, $nk\mu_c$ is 550° which is very close to 3π. The factor $\left[\frac{sinnk\mu_c}{\sin k\mu_c}\right]^2$ becomes 3.7 instead of 100, which annihilates the effect of the sextupole families in the vertical plane. This was noticed at the time of the first LEP study and non periodic sextupole families were used to correct the non-linear chromaticity of this lattice.

A second good example is that of the second LEP lattice [8]. The horizontal phase advance per cell was 60° and the vertical one was 62.1°. The latter is closer to 60° than in the first design, which made it possible to use periodic families with five sextupoles per family (12 families total).

These cases of phase advances per cell slightly different from $60°$ for the case of 10 sextupole per family is instructive. For phase of 57.3° or 62.7°, a factor two is lost in the contribution of the sextupole families. This makes sextupoles increments twice as large for correcting the same effect. Consider a case where the increments of the sextupole strength of the family which has to be increased are about 30% with a $60°$ phase advance. For 57.3°, they have to be increased by 30% more, which makes the dynamic aperture decrease substantially. For the case of 55° quoted above, the factor $\frac{100}{3.7}$ is so large that the correction of the non-linear chromaticity becomes marginal even for a large increase of the sextupole strengths, with the consequence that the dynamic aperture becomes dramatically low [7].

C. Number of sextupoles per family

From formula (5), it is clear that the larger the number of sextupoles per family, the smaller the tolerance on the phase advance per cell to make the correction of the non-linear chromaticity possible. For instance, for 100 sextupoles per family and a phase advance per cell close to $60°$, the efficiency, defined for instance by the ratio $\frac{\sin n\mu_c}{n \sin \mu_c}$, goes to zero for a phase advance per cell of 59.4°. Such a tight tolerance can be avoided by increasing arbitrarily the total number of sextupole families in the case where the number of cells per superperiod is large.

On the opposite, for a small number of cells, the tolerance on the phase advance is much relaxed. For three sextupoles per family, the efficiency of the system looses is reduced by 10% for a phase of about 55°!

V. Sextupole families for out of phase cells

For a case where there it is absolutely necessary to have a phase advance per cell incompatible with periodic families, non-periodic families can be a solution. The best example known to the author is that of the first LEP design [7] quoted above. For this case, the families arrangement is like :

1 2 x 1 2 2 1 x 2 1 1 2 3 1 2 3 1 3 2 1 3 ....

where 1 to 3 refer to the family number and x to a missing sextupole. Such an arrangement has been obtained by inspecting the modulation of the β_y function at the sextupole location on an off-momentum closed orbit and assigning the sextupoles with the same modulation to the same family. An additional rule to obtain a satisfactory system is to make pairs of sextupoles separated by about π phase advance, in order not to produce too much geometric aberrations. It is clear that such a system works only for a given phase advance once it is built, which reduces the lattice flexibility.

VI. Conclusion

Correcting the non-linear chromaticity with sextupole families is easy and powerful when a machine is designed with a number of regular cells having a phase advance equal to an odd number of $\frac{\pi}{k}$ where k is any integer. On top of the designed correction, they provide a simple knob to adjust experimentally the second order tune derivative. Depending on the number of sextupoles per super-period, there is more or less flexibility for changing the machine tune by means of the quadrupoles in the regular cells. The best lattice design to fully exploit the the potentialities of sextupole families is always to make tunable insertions to avoid changing the machine tune with the quadrupoles in the regular cells.

With the phase advance per cell chosen as specified above, the second order geometric aberrations are automatically zero provided there is an even number of sextupoles per family and the same phase advance in both planes [9]. The remaining problem is then the anharmonicities. From the experience with LEP, this is a serious problem only for strong focusing lattices. Some examples can be found at this conference [10].

References

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