ANALYTIC SECOND- AND THIRD-ORDER ACHROMAT DESIGNS

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I. INTRODUCTION

An achromat is a transport system that carries a beam without distorting its transverse phase space distribution. In this study, we apply the Lie algebraic technique [1-6] to a repetitive FODO array to make it either a second-order or a third-order achromat. (Achromats based on reflection symmetries [7,8] are not studied here.) We will consider third-order achromats whose unit FODO cell layout is shown in Fig.1. The second-order achromat layout is the same except the octupoles are absent.

For the second-order achromats, correction terms (due to the finite bending of the dipoles) to the well-known formulae for the sextupole strengths are derived. For the third-order achromats, analytic expressions for the five octupole strengths are given. The quadrupole, sextupole and octupole magnets are assumed to be thin-lens elements. The dipoles are assumed to be sector magnets filling the drift spaces. More details of the analysis have been reported elsewhere.[9] We thank Y. Yan, H. Ye, J. Irwin and A. Dragt for their help.

II. ANALYSIS

We first calculate the Lie maps of each of the magnet elements. The map for a magnet element of length L is gven by $e^{-L:H:}$, where H is the Hamiltonian of the element. For a particle with $\delta = \Delta P/P_0$, we use (we ignore the path-length dynamics)

$$\text{thin quadrupole}: HL = \frac{1}{2F_k}(x^2-y^2)(1-\delta+\delta^2)$$

$$\text{thin sextupole}: HL = \frac{S_k}{3}(x^3-3xy^2)(1-\delta)$$

$$\text{thin octupole}: HL = \frac{O_k}{4}(x^4-6x^2y^2+y^4)$$

$$\text{sector dipole}: H = \frac{P_x^2+P_y^2}{2}+\frac{x^2}{2R^2}-\frac{x\delta}{R}+\frac{x(P_x^2+P_y^2)}{2R}$$
$$-\frac{x^2\delta}{2R^2}+\frac{x\delta^2}{R}+\frac{(P_x^2+P_y^2)^2}{8}-\frac{x\delta^3}{R}+\frac{x^2\delta^2}{2R^2} \qquad (1)$$

where R is the bending radius; F_k is the focal length of the k-th quadrupole; S_k and O_k are the k-th integrated sextupole and octupole strengths. Fringe fields are ignored.

Given the Hamiltonian H of an element, we factorize the element map as

$$e^{-L:H:} = e^{:H_2+H_3+H_4+\cdots:} = e^{:f_2:}e^{:f_3:}e^{:f_4:}e^{:\mathcal{O}(X^5):} \qquad (2)$$

where H_n and f_n are polynomials of order n in the variables $X = (x, P_x, y, P_y, \delta)$. We performed this factorization [3,5] and obtained

$$\text{thin quadrupole}:$$
$$f_3 = \frac{1}{2F_k}(x^2-y^2)\delta, \quad f_4 = -\frac{1}{2F_k}(x^2-y^2)\delta^2$$

$$\text{thin sextupole}:$$
$$f_3 = -\frac{S_k}{3}(x^3-3xy^2), \quad f_4 = \frac{S_k}{3}(x^3-3xy^2)\delta$$

$$\text{thin octupole}: \quad f_3 = 0, \quad f_4 = -\frac{O_k}{4}(x^4-6x^2y^2+y^4)$$

$$\text{sector dipole}:$$
$$\begin{aligned}
f_3 = & -\frac{1}{6R^2}\sin^3\frac{L}{R}\,x^3 - \frac{1}{4R}\sin\frac{L}{R}\sin\frac{2L}{R}\,x^2P_x \\
& -\frac{1}{4}\cos\frac{L}{R}\sin\frac{2L}{R}\,xP_x^2 + \frac{R}{6}(1-\cos^3\frac{L}{R})P_x^3 \\
& -\frac{1}{2}\sin\frac{L}{R}\,xP_y^2 + \frac{x^2\delta}{2R}\sin\frac{L}{R}(\cos\frac{L}{R}+\sin^2\frac{L}{R}) \\
& +R\sin^2\frac{L}{2R}\,P_xP_y^2 - 2\sin^2\frac{L}{2R}\sin^2\frac{L}{R}\,xP_x\delta \\
& -\frac{R}{2}\sin^2\frac{L}{2R}\sin\frac{2L}{R}\,P_x^2\delta - \frac{1}{2}(L-R\sin\frac{L}{R})P_y^2\delta \\
& -\frac{x\delta^2}{2}(\sin^3\frac{L}{R}+\sin\frac{2L}{R}) + \frac{R}{2}(2-\cos\frac{L}{R})\sin^2\frac{L}{R}\,P_x\delta^2 \\
& +\frac{1}{12}(-6L+2R\sin^3\frac{L}{R}+3R\sin\frac{2L}{R})\delta^3
\end{aligned}$$

$$\begin{aligned}
f_4 = & \Big[-\frac{x^2}{8R}\sin^3\frac{L}{R} - \frac{xP_x}{8}\sin\frac{L}{R}\sin\frac{2L}{R} \\
& -\frac{R}{8}\cos^2\frac{L}{R}\sin\frac{L}{R}\,P_x^2 - \frac{R}{8}\sin\frac{L}{R}\,P_y^2\Big](P_x^2+P_y^2) \\
& +\frac{x^3\delta}{12R^2}\sin^3\frac{L}{R} + (\frac{1}{2}+\cos\frac{L}{R})\sin^2\frac{L}{2R}\sin\frac{L}{R}\,xP_x^2\delta \\
& +\Big[\frac{R}{12}(3+4\cos\frac{L}{R}+5\cos\frac{2L}{R})\,P_x^3\delta \\
& +\frac{xP_y^2\delta}{4}\sin\frac{2L}{R} + \frac{R}{4}(3+\cos\frac{2L}{R})P_xP_y^2\delta\Big]\sin^2\frac{L}{2R} \\
& -\frac{1}{4R}(\sin^3\frac{L}{R}+\sin\frac{2L}{R})\,x^2\delta^2 + \frac{1}{2}\sin^2\frac{L}{R}\,xP_x\delta^2 \\
& -\frac{R}{4}(1+3\cos\frac{L}{R})\sin^2\frac{L}{2R}\sin\frac{L}{R}\,P_x^2\delta^2 \\
& +\Big[\frac{R}{2}\sin^4\frac{L}{2R}P_y^2 + (\cos\frac{L}{R}+\frac{1}{4}\sin^2\frac{L}{R})\,x\delta\Big]\sin\frac{L}{R}\delta^2 \\
& -\frac{R}{2}\sin^2\frac{L}{R}\,P_x\delta^3 + \frac{\delta^4}{12}(6L-R\sin^3\frac{L}{R}-3R\sin\frac{2L}{R})
\end{aligned} \qquad (3)$$

Having factorized the maps of all magnets, the total map $\mathcal{M}_{\text{cell}}$ of a cell is obtained by multiplying and concatenating the maps of the component elements:[3,9]

$$\mathcal{M}_{\rm cell} = \prod_{i=1}^{N}(e^{:f_2^i:}e^{:f_3^i:}e^{:f_4^i:}) = e^{:h_2:}e^{:h_3:}e^{:h_4:}e^{:\mathcal{O}(X^5):} \tag{4}$$

where

$$\mathcal{R} = e^{:h_2:} = \prod_{i=1}^{N} e^{:f_2^i:}, \qquad h_3 = \sum_{i=1}^{N}\tilde{f}_3^i$$
$$h_4 = \sum_{i=1}^{N}\tilde{f}_4^i + \frac{1}{2}\sum_{j>i=1}^{N}[\tilde{f}_3^i, \tilde{f}_3^j] \tag{5}$$

In Eq.(5), $\tilde{f}^i$ means $\tilde{f}^i(X) = f^i(R_{N\to i}X)$ with $R_{N\to i}$ the linear map from the last element to the i-th element. The map of the N-cell achromat is $\mathcal{M} = \mathcal{M}_{\rm cell}^N$. The number of cells N is so that $\mu_{x,y}$ (the total phase advances in x are y) are both multiples of 2π, but avoiding resonances.

We now make a canonical coordinate transformation from (x, P_x, y, P_y) to $(\phi_x, A_x, \phi_y, A_y)$ by $x = \sqrt{2A_x\beta_x}\sin\phi_x + \eta\delta$, $P_x = \sqrt{\frac{2A_x}{\beta_x}}(\cos\phi_x - \alpha_x\sin\phi_x) + \eta'\delta$, and similarly for y and P_y without the η and η' terms, where $\beta_{x,y}, \alpha_{x,y}$ and η, η' are the Courant-Snyder and the dispersion functions.[10] The linear map generator h_2 becomes $h_2 = -\mu_x A_x - \mu_y A_y - \frac{1}{2}\bar{\alpha}_c\delta^2$ where and $\bar{\alpha}_c$ is the momentum compaction factor. We then decompose h_n in terms of the eigenmodes of $:h_2:$ as[5]

$$h_n = \sum_{a+b+c+d+e=n} C^n_{abcd,e}|abcd,e\rangle$$
$$|abcd,e\rangle \equiv A_x^{(a+b)/2}A_y^{(c+d)/2}e^{i(a-b)\phi_x}e^{i(c-d)\phi_x}\delta^e \tag{6}$$

To reduce a nonlinear map to its normal form, it can be shown[11] that (in the absence of resonances)[2] all the nonsecular terms can be transformed away via a symplectic similarity transformation leaving only terms with $a = b$ and $c = d$, i.e., terms depending on A_x, A_y and δ only. In particular, we have

$$h_3 = C^3_{1100,1}A_x\delta + C^3_{0011,1}A_y\delta + C^3_{0000,3}\delta^3$$
$$h_4 = C^4_{2200,0}A_x^2 + C^4_{0022,0}A_y^2 + C^4_{1111,0}A_xA_y$$
$$+C^4_{1100,2}A_x\delta^2 + C^4_{0011,2}A_y\delta^2 + C^4_{0000,4}\delta^4 \tag{7}$$

III. SECOND-ORDER ACHROMATS

For a second-order achromat, we follow Eqs.(6-7) and find the normal form of the unit cell is given by h_3 of Eq.(7) where

$$C^3_{1100,1} = \sum_{k=1,2}^{\rm quads}\left[\frac{1}{2F_k} - \lambda_k\eta(k)\right]\beta_x(k) + w_x$$
$$C^3_{0011,1} = -\sum_{k=1,2}^{\rm quads}\left[\frac{1}{2F_k} - \lambda_k\eta(k)\right]\beta_y(k) + w_y \tag{8}$$

and

$$w_x = \sum_{k=1,2}^{\rm dipoles}\frac{1}{2}\sin^2(\frac{L}{R})\Big\{\frac{\beta_x(k)}{R}\Big[\sin\frac{L}{R} + \cot\frac{L}{R}$$
$$-\frac{\eta(k)}{R}\sin\frac{L}{R} - \eta'(k)\cos\frac{L}{R}\Big] + 2\alpha_x(k)\Big[1 - \cos\frac{L}{R}$$
$$+\frac{\eta(k)}{R}\cos\frac{L}{R} + \eta'(s)\cos\frac{L}{R}\cot\frac{L}{R}\Big]$$
$$+\gamma_x(k)R\Big[-\cos\frac{L}{R}\tan\frac{L}{2R} - \frac{\eta(k)}{R}\cos\frac{L}{R}\cot\frac{L}{R}$$
$$+(\cos\frac{L}{R} + \frac{1}{2}\sec^2\frac{L}{2R})\eta'(k)\Big]\Big\}$$
$$w_y = \sum_{k=1,2}^{\rm dipoles}\frac{1}{2}\gamma_y(k)R\Big[\sin\frac{L}{R} - \frac{L}{R} - \eta(k)\sin\frac{L}{R}$$
$$+\eta'(k)(1 - \cos\frac{L}{R})\Big] \tag{9}$$

The lattice functions are evaluated at the two quadrupoles in Eq.(8) and at the ends of the two dipoles in Eq.(9). In the limit of weak bending with $\epsilon_1 = \frac{L}{R} \ll 1$, we have

$$w_x \simeq \epsilon_1\sum_s^D \alpha_x(s)\eta'(s) + \frac{1}{4}\gamma_x(s)(3L\eta'(s) - 2\eta(s))$$
$$w_y \simeq \epsilon_1\sum_s^D \frac{1}{4}\gamma_y(s)(L\eta'(s) - 2\eta(s)) \tag{10}$$

To form a second-order achromat, we set the two C-coefficients to zero, and obtain

$$S_1 = \frac{1}{2\eta(1)F_1} + \frac{\beta_y(2)w_x + \beta_x(2)w_y}{\eta(1)[\beta_x(1)\beta_y(2) - \beta_x(2)\beta_y(1)]}$$
$$S_2 = \frac{1}{2\eta(2)F_2} - \frac{\beta_y(1)w_x + \beta_x(1)w_y}{\eta(2)[\beta_x(1)\beta_y(2) - \beta_x(2)\beta_y(1)]} \tag{11}$$

The first terms usually dominate and give the well-known results. The correction terms with w_x and w_y are normally but not always small.

IV. THIRD-ORDER ACHROMATS

We also studied the case of a third-order achromat. An algebraic program using Mathematica was developed to do the analysis. Here we only report our results. The normal form of the third-order generator for a unit cell is given by Eq.(9) with

$$C^4_{2200,0} = -\frac{3}{8}\sum_{k=1}^{5}\beta_x(k)^2 O_k + w_{xx}$$
$$C^4_{1111,0} = \frac{3}{2}\sum_{k=1}^{5}\beta_x(k)\beta_y(k)O_k + w_{xy}$$
$$C^4_{0022,0} = -\frac{3}{8}\sum_{k=1}^{5}\beta_y(k)^2 O_k + w_{yy}$$
$$C^4_{1100,2} = -\frac{3}{2}\sum_{k=1}^{5}\beta_x(k)\eta(k)O_k + w_{xd}$$
$$C^4_{0011,2} = \frac{3}{2}\sum_{k=1}^{5}\beta_y(k)\eta(k)O_k + w_{yd} \tag{12}$$

and (when $\epsilon_1 = \frac{L}{R} \ll 1$)

$$w_{xx} \simeq \csc\frac{3\mu_x}{2}(2 + 3\cos\mu_x)\prod_s^S\frac{S_s}{4}\beta_x(s)^{\frac{3}{2}} - \frac{3}{16}\frac{L}{}\sum_s^D\gamma_x(s)^2$$
$$+\frac{1}{8}\csc\frac{3\mu_x}{2}(3\cos\frac{\mu_x}{2} + 2\cos\frac{3\mu_x}{2})\sum_s^S S_s^2\beta_x(s)^3$$

$$
\begin{aligned}
w_{xy} \simeq & -\frac{L}{4}\sum_s^D \gamma_x(s)\gamma_y(s) \;-\; \frac{1}{2}\cot\frac{\mu_x}{2}\sum_s^S S_s^2\beta_x(s)^2\beta_y(s) \\
& -\csc(\frac{\mu_x}{2}+\mu_y)\csc(\frac{\mu_x}{2}-\mu_y)\sin 2\mu_y\sum_s^S \frac{S_s^2}{4}\beta_x(s)\beta_y(s)^2 \\
& +\left[\csc(\frac{\mu_x}{2}+\mu_y)-\csc(\frac{\mu_x}{2}-\mu_y)-\csc\frac{\mu_x}{2}\sum_s^S\frac{\beta_x(s)}{\beta_y(s)}\right] \\
& \times\frac{1}{2}\prod_s^S S_s\sqrt{\beta_x(s)}\beta_y(s) \\
w_{yy} \simeq & -\frac{3\,L}{16}\sum_s^D \gamma_y^2(s) \;+\; \frac{1}{16}\sum_s^S S_s^2\beta_x(s)\beta_y(s)^2 \\
& \times\left[4\cot\frac{\mu_x}{2}+\sin\mu_x\csc(\frac{\mu_x}{2}+\mu_y)\csc(\frac{\mu_x}{2}-\mu_y)\right] \\
& +\frac{1}{8}\left[4\csc\frac{\mu_x}{2}+\csc(\frac{\mu_x}{2}+\mu_y)+\csc(\frac{\mu_x}{2}-\mu_y)\right] \\
& \times\prod_s^S S_s\sqrt{\beta_x(s)}\beta_y(s) \\
w_{xd} \simeq & -\frac{3\,L}{4}\sum_s^D \gamma_x(s)\eta'(s)^2 \;-\; \sum_s^S\beta_x(s)(\frac{1}{2F_s}-S_s\beta_x(s)) \\
& +\frac{1}{2}\cot\mu_x\sum_s^S\left[\beta_x(s)(\frac{1}{2F_s}-S_s\beta_x(s))\right]^2 \\
& +\csc\mu_x\prod_s^S\beta_x(s)(\frac{1}{2F_s}-S_s\beta_x(s)) \\
& +\csc\frac{\mu_x}{2}\sum_{1,2}^S\frac{S_2}{2}\beta_x(2)\eta(1)(S_1\eta(1)-\frac{1}{F_1})\sqrt{\beta_x(1)\beta_x(2)} \\
& +\frac{1}{2}\cot\frac{\mu_x}{2}\sum_s^S S_s\eta(s)(S_s\eta(s)-\frac{1}{F_s})\beta_x(s)^2 \\
w_{yd} \simeq & -\frac{L}{4}\sum_s^D \gamma_y(s)\eta'(s)^2 \;+\; \sum_s^S\beta_y(s)(\frac{1}{2F_s}-S_s\beta_x(s)) \\
& +\frac{1}{2}\cot\mu_y\sum_s^S\left[\beta_y(s)(\frac{1}{2F_s}-S_s\beta_x(s))\right]^2 \\
& +\csc\mu_y\prod_s^S\beta_y(s)(\frac{1}{2F_s}-S_s\beta_x(s)) \\
& -\csc\frac{\mu_x}{2}\sum_{1,2}^S\frac{S_2}{2}\beta_y(2)\eta(1)(S_1\eta(1)-\frac{1}{F_1})\sqrt{\beta_x(1)\beta(2)} \\
& -\frac{1}{2}\cot\frac{\mu_x}{2}\sum_s^S S_s\eta(s)(S_s\eta(s)-\frac{1}{F_s})\beta_x(s)\beta_y(s)
\end{aligned}
\tag{13}
$$

Exact expressions of the w-coefficients are too lengthy to be included here.

The required octupole strengths are such that the five C-coefficients in Eq.(12) are equal to zero. For the case when two of the octupoles are located next to the two sextupoles and the other three are at the $\frac{1}{3}$, $\frac{2}{3}$, and the $\frac{1}{2}$ locations of the two bending magnets, we find

$$
\begin{aligned}
& O_1 \simeq \frac{a+b}{6f^3D}, \quad O_2 \simeq \frac{81(c+d)}{2fD}, \quad O_3 \simeq \frac{81(c-d)}{2fD} \\
& O_4 \simeq \frac{a-b}{6f^3D}, \quad O_5 \simeq \frac{128e}{3(2f^2-1)D} \\
& a = 2f(1360-22846f^2-74476f^4+695809f^6 \\
& \quad -1438146f^8+1200096f^{10}-326592f^{12}) \\
& b = -352-3360f^2+233290f^4-1070910f^6 \\
& \quad +1917603f^8-1364850f^{10}+361584f^{12} \\
& c = 6f(-42+1076f^2-7409f^4+16306f^6-14368f^8 \\
& \quad +4032f^{10}) \\
& d = 8-394f^2+5322f^4-16907f^6+14866f^8-4464f^{10} \\
& e = -368+10536f^2-92342f^4+307222f^6-470547f^8 \\
& \quad +330642f^{10}-81648f^{12} \\
& D = (4f^2-1)^2(3f^2-4)(10-173f^2-261f^4+324f^6)L^3\epsilon_1^2
\end{aligned}
\tag{14}
$$

We have defined the dimensionless parameter $f = \frac{2F_1}{L}$ and have assumed that $\epsilon_1 = \frac{L}{R} \ll 1$ and $|\frac{F_1+F_2}{L}| \ll 1$.

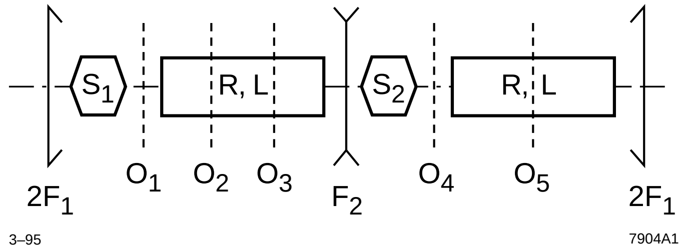

Figure. 1. Unit cell of an achromat layout.

References

[1] K.L. Brown, SLAC-PUB-3381 (1984).

[2] E. Forest, D. Douglas and B. Leemann, Nucl. Instr. Meth. A258, 355 (1987).

[3] A.J. Dragt, AIP conf. Proc. No.87, Phys. high energy part. accel., 1982, p.147.

[4] A.J. Dragt and J.M. Finn, J. Math. Phys. 17, 2215 (1976).

[5] A.J. Dragt and E. Forest, J. Math. Phys. 24, 12 (1983).

[6] A.J. Dragt, Nucl. Instr. Meth. A258, 339 (1987).

[7] W. Wan, E. Goldmann and M. Berz, AIP Proc. 297 (1994), p.143.

[8] W. Wan and M. Berz, Proc. Part. Accel. Conf., 1993, p.155.

[9] Chunxi Wang and Alex Chao, SLAC internal report AP-100 (1995).

[10] E.D. Courant and H.S. Snyder, Ann. Phys. 3,1 (1958).

[11] E. Forest, SSC Report 29 (1985).