THE BEAM ENVELOPE EQUATION - SYSTEMATIC SOLUTION FOR A FODO LATTICE WITH SPACE CHARGE

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ABSTRACT

Many approximate solutions for matched beam envelope functions with space charge have been developed; they generally have errors of 2 - 10% for the parameters of interest and cannot be reliably improved. The new, systematic approach described here provides the K-V envelope functions to arbitrarily high accuracy as a power series in the quadrupole gradient. A useful simplification results from defining the sum and difference of the envelope radii; S = (a+b)/2 varies only slightly with distance z along the system axis, and D = (a-b)/2 contains most of the envelope oscillations. To solve the coupled equations for S and D, the quadrupole strength K(z) is turned on by replacing K with αK_1 and letting α increase continuously from 0 to 1. It is found that S and D may be expanded in even and odd powers of α, respectively. Equations for the coefficients of powers of α are then solved successively by integration in z. The periodicity conditions and tune integration close the calculation. Simple low order results are typically accurate to 1% or better.

I. INTRODUCTION

The matched (periodic) solution of the coupled Kapchinskij-Vladimirskij (K-V) beam envelope equations is used extensively in the design of quadrupole transport systems.(1) Exact results for a specified set of beam and lattice parameters are readily obtained numerically. However to perform scoping studies, scaling, cost optimization, and to gain physical understanding, it is very desirable to have simple analytical formulas for the envelope functions. The general problem may be stated as follows. For specified quadrupole strength K(z) with period (P), beam edge emittance (ε), and beam perveance (Q), find the matched envelope radii a(z) and b(z). The depressed phase advance per period or tune (σ) is then determined from the mean of a^{-2}, and the undepressed phase advance (σ_0) is determined in similar fashion in the limit Q = 0.

The simplest rough design formulas are obtained by assuming that the quadrupoles effectively act like a confining harmonic well with frequency $\sigma_0/2\pi$ P and that the envelope radii are approximated by their mean values ($\bar{a}$). The perveance and emittance are then related to the tunes, σ_0 and σ (expressed in radians) by the “smooth limit” formulas

$$\frac{QP^2}{\bar{a}^2} = \sigma_0^2 - \sigma^2 \quad , \quad \frac{\sigma P}{\bar{a}^2} = \sigma \quad , \qquad (1,2)$$

These formulas are correct in the limit σ_0 ™ 0, but are in error by up to 40% for the typical design case of σ_0 = 72˚. Furthermore the smooth limit, by itself, does not provide formulas for σ_0 , the maximum edge radius, or the envelope slopes. The evaluation of the latter quantities requires a definite prescription for K(z) and treatment of the associated envelope oscillations. A new approach to solving for matched envelope functions is described here. Essentially, an expansion in powers of K^2P^4 is made for various envelope quantities, and the low order non-trivial formulas are typically accurate to about 1% or better. Details are contained in an LBL report by the author(2).

II. THE K-V ENVELOPE EQUATIONS

The x and y radii, a(z) and b(z), are assumed to satisfy the coupled, non-linear envelope equations of Kapchinskij and Vladimirskij:

$$\frac{d^2a}{dz^2} = -K(z)a + \frac{\varepsilon^2}{a^3} + \frac{2Q}{a+b} \quad , \qquad (3)$$

$$\frac{d^2b}{dz^2} = +K(z)b + \frac{\varepsilon^2}{b^3} + \frac{2Q}{a+b} \quad . \qquad (4)$$

Here the quadrupole strength K(z) is the ratio of the transverse magnetic field gradient G(z) and particle rigidity [Bρ] = $\beta\gamma$mc/q. The perveance is the dimensionless constant proportional to current as defined by Lawson: $Q=2qI[(\beta\gamma)^3 mc^3 4\pi\varepsilon_0]^{-1}$, and the (un-normalized) edge emittance $\varepsilon = \varepsilon_x = \varepsilon_y$ is the occupied (x, dx/dz) phase space area divided by π.

By assumption, K(z+P) = K(z) . We also assume the mean of K vanishes, and K is antisymmetric around z=0; K(-z) = -K(z). Denoting the half period length L = P/2, it follows that K(z) is also antisymmetric around z = ± L, ± 2L, ... No additional symmetry is assumed, so a system of unsymmetrical quadrupole doublets is accommodated by the general formulation.

The matched envelope radii exhibit the periodicity of the lattice: a(z+P) = a(z) , b(z+P) = b(z) , and at mid-drift points z = 0, ± L, ± 2L, ..., it is easily shown that a = b , da/dz = -db/dz .

We define the sum and difference of envelope radii: S(z) = (a+b)/2, D(z) = (a-b)/2 . From the symmetries of a and b, it follows that S is symmetric and D is antisymmetric around z = 0, ± L, ± 2L, ... Hence, S´(0) = S´(L) = 0 , D(0) = D(L) = 0.

Next we define the dimensionless envelope functions s(z) and d(z):

$$S(z) = \bar{a}(1 + s(z)) \ , \quad D(z) = \bar{a} \ d(z) \ . \qquad (5a,b)$$

Denoting d/dz by a superscript prime, eqns (3) and (4) are added and subtracted to yield

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$$s'' = -Kd + \frac{\varepsilon^2}{\bar{a}^4}\frac{1}{2}\left[\frac{1}{(1+s+d)^3} - \frac{1}{(1+s-d)^3}\right] + \frac{Q}{\bar{a}^2}\frac{1}{1+s} \quad (6)$$

$$d'' = -K(1+s) + \frac{\varepsilon^2}{\bar{a}^4}\frac{1}{2}\left[\frac{1}{(1+s+d)^3} - \frac{1}{(1+s-d)^3}\right] \quad . \quad (7)$$

The functions s(z) and d(z) satisfy the same symmetry conditions as S and D respectively. Since the envelope radii are periodic, we have the condition

$$\overline{Kd} = \frac{\varepsilon^2}{\bar{a}^4}\frac{1}{2}\left[\overline{\frac{1}{(1+s+d)^3}} + \overline{\frac{1}{(1+s-d)^3}}\right] + \frac{Q}{\bar{a}^2}\overline{\frac{1}{(1+s)}} \quad . \quad (8)$$

In general, the rate of phase advance in the x or y plane is the inverse of the respective β function (defined by $\beta_x = a^2/\varepsilon$). A useful expression for the tune is:

$$\sigma = \left(\frac{P\varepsilon}{\bar{a}^2}\right)\frac{1}{2}\left[\overline{\frac{1}{(1+s+d)^2}} + \overline{\frac{1}{(1+s-d)^2}}\right] \quad . (9)$$

Recall that in the initial formulation of the matched envelope calculation K(z), ε and Q are considered to be specified and the matched radii are to be determined. However in eqns (6) - (9), $\bar{a}$ actually gets absorbed into combinations with ε and Q; only K(z), $\varepsilon/\bar{a}^2$, and $Q/\bar{a}^2$ appear. Due to the matched envelope condition (8), these three quantities cannot be specified independently. Intuitively this is clear since, for example, if we set $\varepsilon = 0$ then the transport is space-charge-dominated and we would expect the current <u>density</u> J , $Q/\bar{a}^2$ to be determined by K(z) alone.

III. METHOD OF SOLUTION

We turn on K(z) proportional to a continuous variable (α) ; $K(z) = \alpha K_1(z)$, where $K_1(z)$ is the full quadrupole strength function and α increases from 0 to 1. For small α, we except to recover the smooth limit formulas. As α increases the envelope radii become lumpy, i.e. d(z) becomes appreciable. As K(z) turns on it is also necessary that $\varepsilon/\bar{a}^2$ and $Q/\bar{a}^2$ turn on. A natural dependence suggested by the smooth limit formulas (1) is $\varepsilon/\bar{a}^2 \sim \alpha$ and $Q/\bar{a}^2 \sim \alpha^2$, so that $\sigma_0 \sim \alpha$ and σ_0/σ is independent of α. Due to the condition (8), it cannot be quite this simple; the system would be overdetermined beyond the lowest two orders in α. A consistent, but not unique, turn-on procedure is to hold the ratio $(Q\bar{a}^2/\varepsilon^2)$ fixed and define the angle (ϕ) : $\cos^2\phi = \left(1 + Q\bar{a}^2/\varepsilon^2\right)^{-1}$. Then convenient forms $\varepsilon/\bar{a}^2$ and $Q/\bar{a}^2$ are

$$\varepsilon^2/\bar{a}^4 = \cos^2\phi\left(A_2\alpha^2 + A_4\alpha^4 + \ldots\right) \quad , \quad (10)$$

$$Q/\bar{a}^2 = \sin^2\phi\left(A_2\alpha^2 + A_4\alpha^4 + \ldots\right) \quad , \quad (11)$$

where A_2, A_4, ... are determined from eqn (8). The consistent expansions for s(z) and d(z) are found to be of the form

$$s(z) = s_2(z)\alpha^2 + s_4(z)\alpha^4 + \ldots \quad , \quad (12)$$

$$d(z) = d_1(z)\alpha + d_3(z)\alpha^3 + \ldots \quad . \quad (13)$$

Inserting the expansions (10) - (13) into eqns (6) and (7), expanding all expressions in powers of α, and equating coefficients of each power of α, we get

$$d_1'' = -K_1(z) \quad , \quad (14)$$

$$s_2'' = -K_1 d_1 + A_2\cos^2\phi + A_2\sin^2\phi \quad , \quad (15)$$

$$d_3'' = -K_1 s_2 - 3A_2 d_1 \cos^2\phi \quad , \quad (16)$$

and so forth. We are now able to solve sequentially for d_1, s_2, d_3, s_4, etc., by a straightforward program of integration. The associated constants (A_n) are determined from averages ($\bar{s}_n'' = 0$) and can always be evaluated using lower order s and d functions. The first two A_n are

$$A_2 = \overline{K_1 d_1} \quad , \quad (17)$$

$$A_4 = \overline{K_1 d_1 s_2} - 3\overline{K_1 d_1}\,\overline{d_1^2}\cos^2\phi \quad . \quad (18)$$

For most applications $\sigma_0 \le 90^\circ$, and it is found that $|d_5| << 01$ and $|s_4| << .001$, so they are not included in further calculations here. Typically $|s_2| \approx .01$, and $|d_3|$ increases from .01 to .05 as cos ϕ ™ 1. Although the formalism developed so far is self-contained, it is of interest to calculate the tunes σ and σ_0 associated with the matched envelopes. The expansion of eqn (9) in powers of α yields

$$\frac{\sigma\bar{a}^2}{P\varepsilon} = 1 + \overline{3d_1^2}\alpha^2 + \left(\overline{3s_2^2} + \overline{6d_1d_3} - \overline{12s_2d_1^2} + \overline{5d_1^4}\right)\alpha^4 + \ldots \quad .(19)$$

The quantity $\varepsilon/\bar{a}^2$ may be eliminated using eqn (10) to obtain

$$\sigma^2 = P^2\cos^2\phi\left[\overline{K_1d_1}\alpha^2 + \overline{K_1d_1s_2}\alpha^4 + 3\overline{K_1d_1}\,\overline{d_1^2}\left(2 - \cos^2\phi\right)\alpha^4 + \ldots\right] \quad . \quad (20)$$

Improved convergence is obtained for the expansion of cos σ as compared with σ^2. Similarly, the expansion of $(\bar{a}^2\sin\sigma)/P\varepsilon$ converges more rapidly than that of $\bar{a}^2\sigma/P\varepsilon$. This behavior of expansions is not surprising because the trigonometric functions of σ appear in the full period transfer matrix. We evaluate the expansions

$$\cos\sigma = 1 - \frac{\sigma^2}{2} + \frac{\sigma^4}{24} - \ldots$$

$$= 1 - \left(\frac{P^2\,\overline{K_1d_1}\cos^2\phi}{2}\right)\alpha^2 + \left[\frac{P^4\left(\overline{K_1d_1}\right)^2\cos^4\phi}{24} - \frac{P^2}{2}\overline{K_1d_1s_2}\cos^2\phi - \frac{3}{2}P^2\,\overline{K_1d_1}\,\overline{d_1^2}(2-\cos^2\phi)\cos^2\phi\right]\alpha^4 + \ldots ,(21)$$

$$\frac{\bar{a}^2\sin\sigma}{P\varepsilon} = \left(\frac{\bar{a}^2\sigma}{P\varepsilon}\right)\left(1 - \frac{\sigma^2}{6} + \ldots\right)$$

$$= 1 + \left(\overline{3d_1^2} - \frac{P^2\,\overline{K_1d_1}}{6}\cos^2\phi\right)\alpha^2 + \ldots \quad . \quad (22)$$

The undepressed tune (σ_0) is obtained from eqn (21) by setting cos ϕ equal to unity. Note that d_1 and s_2 do not depend on ϕ, so that terms of cos σ_0 through (α^4) may be immediately written down:

$$\cos\sigma_0 = 1 - \frac{P^2\overline{K_1 d_1}}{2}\alpha^2 + \left[\frac{P^4\left(\overline{K_1 d_1}\right)^2}{24} - \frac{P^2\overline{K_1 d_1 s_2}}{2} - \frac{3P^2\overline{K_1 d_1}\,\overline{d_1^2}}{2}\right]\alpha^4 + \ldots \quad (23)$$

Unfortunately, the expansion of tunes in powers of α becomes very cumbersome beyond the lowest non-trivial order. Some simplification is achieved by combining formulas in such a way that some of the higher order terms cancel. A spectacular cancellation of terms may be verified for the combination

$$2(\cos\sigma - \cos\sigma_0) - \left(\frac{QP^2}{\bar{a}^2}\right) + \left(\frac{QP^2}{\bar{a}^2}\right)\left[\frac{\cos\sigma - \cos\sigma_0}{6} - \left(\frac{\bar{a}^2\sin\sigma}{P\varepsilon} - 1\right)\right] = O(\alpha^6) \ . \quad (24)$$

It is recommended that this equation be used in place of eqn (21).

IV. FLAT TOP QUADRUPOLE FIELDS

A very useful set of design formulas is derived for the simple FODO lattice with drifts of length $(1-\eta)L$ centered at $z = 0, \pm L, \pm 2L, \ldots$, and flat-topped quadrupoles of strength $\pm k$ and length ηL centered at $z = \pm L/2, -3L/2, \ldots$.

In the interval $0 < z < L/2$, we have

$$d_1'' = \begin{cases} 0 & 0 < z < (1-\eta)\frac{L}{2} \ , \\ -k & (1-\eta)\frac{L}{2} < z < \frac{L}{2} \ . \end{cases} \quad (25)$$

Integrating eqn (14) twice yields

$$d_1 = \begin{cases} \frac{k\eta L}{2} z & 0 < z < (1-\eta)\frac{L}{2} \ , \\ \frac{\eta k L^2}{4}\left(1 - \frac{\eta}{2}\right) - \frac{k}{2}\left(\frac{L}{2} - z\right)^2 & (1-\eta)\frac{L}{2} < z < \frac{L}{2} \ . \end{cases} \quad (26)$$

The maximum value of d_1 is

$$d_{1m} = d_1(L/2) = \frac{\eta k L^2}{4}\left(1 - \frac{\eta}{2}\right) \ , \quad (27)$$

and the useful averages over d_1 are

$$\overline{K_1 d_1} = \frac{\eta^2 k^2 L^2}{4}\left(1 - \frac{2}{3}\eta\right) \ , \quad (28)$$

$$\overline{d_1^2} = \frac{\eta^2 k^2 L^4}{48}\left(1 - \eta^2 + \frac{2}{5}\eta^3\right) \ . \quad (29)$$

Equation (15) is also integrated twice, making use of the condition $\overline{s_2} = 0$, to obtain the minimum and maximum values

$$s_2(0) = -\frac{\eta^2 k^2 L^4}{24}\left(\frac{1}{4} - \frac{\eta}{6} - \frac{\eta^2}{4} + \frac{\eta^3}{5}\right) \ , \quad (30)$$

$$s_2(L/2) = \frac{\eta^2 k^2 L^4}{24}\left(\frac{1}{2} - \frac{13}{12}\eta + \frac{13}{16}\eta^2 - \frac{\eta^3}{5}\right) \ . \quad (31)$$

A double integration of eqn (19) gives the maximum of $d_3(z)$:

$$d_3(L/2) = \frac{\eta^3 k^3 L^6}{192}\left(1 - \frac{19}{6}\eta + \frac{463}{120}\eta^2 - \frac{511}{240}\eta^3 + \frac{9}{20}\eta^4\right) + \frac{\eta^3 k^3 L^6}{64}\left(1 - \frac{2}{3}\eta\right)\left(1 - \frac{\eta^2}{2} + \frac{\eta^3}{8}\right)\cos^2\phi \ . \quad (32)$$

Using the averages $\overline{k_1 d_1}$, $\overline{d_1^2}$, and $\overline{k_1 d_1 s_2}$, we get the tune formulas from eqns (22) and (23):

$$2(1 - \cos\sigma_0) = \eta^2 k^2 L^4\left(1 - \frac{2\eta}{3}\right) - \eta^4 k^4 L^8\left(\frac{\eta^2}{90} - \frac{\eta^3}{63} + \frac{\eta^4}{180}\right) + \ldots \ , \quad (33)$$

$$\left(\frac{\bar{a}^2\sin\sigma}{2L\varepsilon} - 1\right) = \eta^2 k^2 L^4\left[\frac{1}{16}\left(1 - \eta^2 + \frac{2}{5}\eta^3\right) - \frac{\cos^2\phi}{6}\left(1 - \frac{2}{3}\eta\right) \ . \quad (34)$$

Equation (24), relating Q to other parameters, does not depend on the specific form of $K_1(z)$ in the order of approximation included here and is therefore not repeated in this section.

REFERENCES

[1] I.M. Kapchinskij and V.V. Vladimirskij, Proc. Intern. Conf. on High Energy Accelerators, CERN, Geneva, 1959, p. 274.

[2] E. Lee, LBL-37050, April 1995.