EXACT MAP THROUGH IDEAL BENDS (AGAIN?)

Leo Michelotti

Fermi National Accelerator Laboratory[*]

P. O. Box 500, Batavia, IL 60510, USA

Abstract

There are three logically independent facets to calculating the transfer map through a bend magnet: physics, geometry, and representation. We will derive the exact map for transit through ideal bends while separating these three, esp., isolating the geometry problem from the other two.

I. INTRODUCTION

Writing the exact transfer map through ideal bend magnets requires considerations of physics, geometry, and the particular representation in which a particle's state is expressed. Although logically independent, these are frequently mixed together. We will attempt to separate them. It is likely that many people have already gone through this exercise for themselves, but it may be worth repeating.[1]

II. PHYSICS

We all learned the relevant physics as undergraduates. A charged particle in a constant magnetic field, $\vec{B}$, travels at constant speed on a helix aligned along the field. Since the particle experiences no acceleration in the direction of $\vec{B}$,

$$z_\|(t_f) = z_\|(t_i) + v_\| \Delta t \ , \tag{1}$$

where the subscript "$\|$" stands in for the appropriate coordinate projection, and $\Delta t = t_f - t_i$ is the time spent in the magnetic environment. The (radial) frequency of travelling around the helix is,

$$\omega = |\, eB/\gamma m \,| \ ,$$

in rationalized mks units, where, e is the charge of the particle, m is its mass, and $\gamma = 1/\sqrt{1-\beta^2} = E/mc^2$ is the usual relativistic factor. Projected onto a plane orthogonal to $\vec{B}$, the helical orbit becomes a circle of radius ρ,

$$\rho = |\, p_\perp/eB \,| \ ,$$

where $p_\perp$ is the projection of the particle's momentum onto the plane orthogonal to $\vec{B}$.

This is all we need. In the sections to follow we will complete the derivation by (a) solving a few elementary geometry problems and (b) writing the answers in the accelerator physicists' representation.

III. GEOMETRY

The natural Cartesian chart for expressing the geometry, which we will call the "Z-chart," has its origin on the helical axis with one coordinate axis parallel to it. Following a standard convention, for bending magnets we choose the z_2-axis parallel to $\vec{B}$, while for solenoids, we would align z_3 along the $\vec{B}$, so that in both cases the largest component of momentum is along the z_3 axis. (Please refure to Figures 1 through 3 repeatedly for visualization of the charts in this paper.) Because it is easiest to work with the transverse equations in terms of spinors, we define complex coordinates $z \equiv z_3 + iz_1$, for bend magnets, or $z \equiv z_1 + iz_2$, for solenoids, and write the transversely projected dynamics in either case as follows.[2]

$$z = \rho e^{i\theta} \ , \ \theta \in [-\pi, \pi) \ , \tag{2}$$
$$\dot{z} = -i\omega z \ . \tag{3}$$

The time taken to cross the magnetic environment is clearly $\Delta t = -\Delta\theta/\omega$, and we will address below the purely geometric problem of calculating $\Delta\theta$.

A difficulty arises because the orbit is not viewed from the Z-chart but from two local charts, say U_i and U_f, which we will call the "initial" and "final" charts, or, alternatively, the "in" and "out" charts. Although these are, in principle, arbitrary, for practical applications, each typically has an axis aligned along $\vec{B}$: u_2 for bends, and u_3 for solenoids.[3] In the treatment that follows we will handle only the bend-magnet configuration explicitly, leaving the easier problem of solenoids to the reader.

For bending magnets, then, the (u_1, u_2) plane (i.e., $u_3 = 0$) is considered the "transverse plane" or "face" of the chart, called so because it is usually considered to be transverse to the local fiducial reference curve, not to the magnetic field. Normally one thinks of the in-face and out-face as located at the edges of the magnet, but this need not be the case; they could be within the body. All that is necessary is that in traversing the region between the faces, the particle is exposed to the environment of a constant magnetic field. We need to establish chart coordinates for two events, *viz.*, the orbit's intersecting the in- and out-faces, the transverse planes of U_i and U_f. The function which converts coordinates of the first event, on U_i, to those of the second, on U_f, is the transfer map for the region.

The time Δt appearing in Eq. 2 is nothing but the time interval between these events,

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[1] The point of doing something this simple is to be reminded that it *is* simple.

[2] This definition does have the disadvantage of making $\dot{\theta}$ negative in Eq. 3, and, later, $\Delta\theta = \theta_f - \theta_i$ negative rather than positive. I've chosen this rather than the less awkward $z \equiv z_1 + iz_3$ for three reasons, not one of which is particularly compelling: (a) it preserves the cyclic ordering of the (z_1, z_2, z_3) triplet, (b) at least in the U.S., accelerator physicists align the "3" axis along the (fiducial) beam current and the "1" axis outward, (c) being at Fermilab, I am accustomed to thinking of beam current as circulating clockwise, and (d) physicists think of time dependence as $e^{-i\omega t}$.

[3] This restriction could be removed, but the resulting equations would then have to be solved numerically. In any case, it would not conform to any practical application.
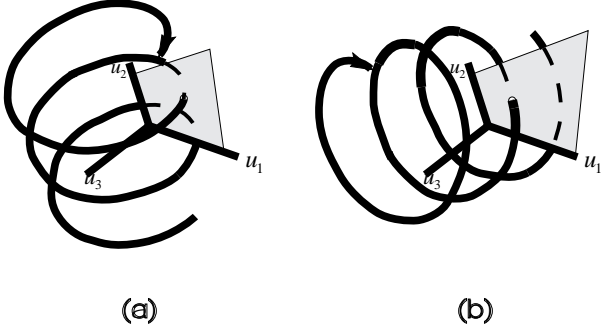
Figure 1. Helical orbit in a constant magnetic field, as viewed from a local chart in (a) a bend magnet and (b) a solenoid.
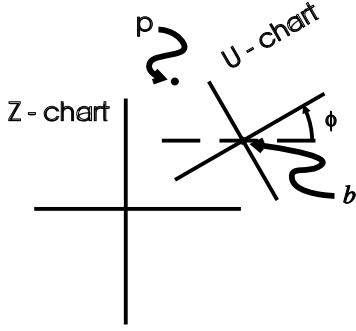

Figure 2. Complex parameters b and ϕ.

Since motion along $\vec{B}$ is taken care of, we can confine our attention to the projection of the orbit on a plane orthogonal to $\vec{B}$ in order to calculate Δt and everything else that we need. The geometric problem we must solve requires nothing more complicated than representing circles and lines in a plane. The transformation which takes us from a U-chart to the Z-chart is written easily in the spinor notation, using $u \equiv u_3 + iu_1$. Referring to Figure 2, we have

$$
\begin{aligned}
z &= b + ue^{i\phi} \ , \quad &(4) \\
\dot{z} &= \dot{u}e^{i\phi} \ , \quad &(5)
\end{aligned}
$$

where b is the Z-chart spinor coordinate of the U-chart's origin, and ϕ is the angle of the u_3 axis relative to the z_3. Our task now is to represent the very simple motion embodied in Eq. 3 on the U charts and thereby obtain the transfer map $U_i \to U_f$. This proceeds in two steps, each of which is a simple geometrical problem: (A) obtaining $be^{-i\phi}$ given u and $\dot{u}$, and (B) finding the intersection of a line with a circle.

A. What is $be^{-i\phi}$?

First, using Eq. 4 and Eq. 3, we have

$$
\dot{z} = -i\omega z = -i\omega\left(b + ue^{i\phi}\right) \ .
$$

Combining this with Eq. 5 yields

$$
\dot{u} = \dot{z}e^{-i\phi} = -i\omega u - i\omega be^{-i\phi} \ ,
$$

from which we obtain $be^{-i\phi}$.

$$
be^{-i\phi} = i\dot{u}/\omega - u \quad (6)
$$

B. Point of intersection between a line and a circle

We must solve for the point of intersection between the u_1 axis of a U-chart and the orbit projection, which is the circle written in Eq. 2. An arbitrary line $L(z_o, \eta)$ passing through a point z_o with direction η is the subset,

$$
L(z_o, \eta) = \{z_o + we^{i\eta} \mid w \in R\}
$$

We easily solve for its intersection with a circle as follows.

$$
\begin{aligned}
z_o + we^{i\eta} &= \rho e^{i\theta} \\
w &= e^{-i\eta}\left(\rho e^{i\theta} - z_o\right)
\end{aligned}
$$

Of course, w must be real, so that, by setting $\text{Im}\,w = 0$, we can find θ.

$$
\sin(\theta - \eta) = \frac{1}{\rho}\text{Im}(z_o e^{-i\eta}) \quad (7)
$$

Applying this to a U-chart, we use the line $L(b, \phi + \pi/2)$; that is, L lies along the imaginary axis of the spinor u. In such cases Eq. 7 becomes,

$$
\theta = \phi + \pi/2 - \arcsin\left(\frac{1}{\rho}\text{Re}(be^{-i\phi})\right), \quad (8)
$$

when $\dot{u}$ is attached to $L(b, \phi + \pi/2)$. We can write this yet another way by applying Eq. 6 to u and $\dot{u}$ evaluated along $L(b, \phi + \pi/2)$. Thus, u will be pure imaginary, so that

$$
\theta = \phi + \pi/2 + \arcsin(\text{Im}(\dot{u})/|\dot{u}|) \ .
$$

IV. OF MAGNETS AND MAPS

We are now ready to complete the construction of a transfer map. Of course, on the Z-chart the "initial" and "final" spinor coordinates are trivially related by a phase rotation.

$$
z_f = z_i e^{i\Delta\theta}, \quad \dot{z}_f = \dot{z}_i e^{i\Delta\theta} \ . \quad (9)
$$

Eq. 4 and Eq. 5 can now be used to obtain u_f and $\dot{u}_f$.

$$
\begin{aligned}
\dot{u}_f &= \dot{u}_i e^{i(\Delta\theta - \Delta\phi)} \quad &(10) \\
u_f &= u_i e^{i(\Delta\theta - \Delta\phi)} + (be^{-i\phi})_i\, e^{i(\Delta\theta - \Delta\phi)} \\
&\quad - (be^{-i\phi})_f \quad &(11)
\end{aligned}
$$

The complete transfer map, $(u_i, \dot{u}_i) \mapsto (u_f, \dot{u}_f)$ is now constructed according to the following procedure.

Bend algorithm

Step 1. Use Eq. 6 to evaluate $(be^{-i\phi})_i$.

$$
(be^{-i\phi})_i = i\dot{u}_i/\omega - u_i.
$$

Step 2. Given $(be^{-i\phi})_i$, from Step 1, construct $(be^{-i\phi})_f$ using the relative in-face to out-face geometry. We will illustrate below how to do this.

Step 3. Calculate $\Delta\theta - \Delta\phi$ by applying Eq. 8 to both faces.

$$
\begin{aligned}
\Delta\theta - \Delta\phi &= \arcsin\left(\text{Re}(be^{-i\phi})_i/\rho\right) \\
&\quad - \arcsin\left(\text{Re}(be^{-i\phi})_f/\rho\right) \quad (12)
\end{aligned}
$$

Step 4. Finally, use Eqs.10 and 11 to complete the map. Notice that $\Delta\theta - \Delta\phi$ appears only in the argument of an exponent. Rather than use Eq. 12 directly we can employ

$$
e^{i(\Delta\theta - \Delta\phi)} = \frac{\sqrt{1 - (\text{Re}(be^{-i\phi})_i/\rho)^2} + i\text{Re}(be^{-i\phi})_i/\rho}{\sqrt{1 - (\text{Re}(be^{-i\phi})_f/\rho)^2} + i\text{Re}(be^{-i\phi})_f/\rho} \ .
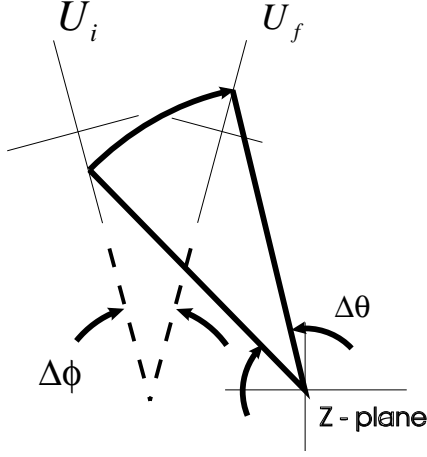$$

Figure 3. Relationships between the U_i, U_f, and Z charts.

A. Rectangular bends

Detailed information about the magnet's geometry, embodied in the relative placements of the in- and out-faces, is used in Step 2. The simplest possibility is the rectangular bend. We take the "in" and "out" faces to be parallel, so that $\phi_i = \phi_f \equiv \phi$. From the obvious relation, $b_f = b_i + Le^{i\phi}$, where L is the length of the magnet, we obtain the result,

$$(be^{-i\phi})_f - (be^{-i\phi})_i = L \ , \tag{13}$$

which is to be fed directly into Step 3.

B. Sector bends

Sector bends are only a little more complicated but easily handled by representing, on the Z chart, the point of intersection between the imaginary axes of the U_i and U_f charts. To position these, we place the face of each orthogonal to a local, fiducial path that would be followed by an "ideal" particle. Let $\bar\rho$ symbolize its radius of curvature. Further, placing the origins of the U-charts on the fiducial path the intersection point has coordinate $-i\bar\rho$ on each chart. Thus, using Eq. 4, we have,

$$(be^{-i\phi})_f - (be^{-i\phi})_i = (1 - e^{-i\Delta\phi})(i\bar\rho - (be^{-i\phi})_i) \ . \tag{14}$$

We note in passing the usual relation between $\Delta\phi$, $\bar\rho$, and magnet length, L, recalling that, by our convention, $\Delta\phi < 0$,

$$e^{-i\Delta\phi} = \left(\sqrt{1 - (L/2\bar\rho)^2} + iL/2\bar\rho \right)^2 \ . \tag{15}$$

Keeping L fixed and letting $\bar\rho \to \infty$ then reproduces Eq. 13.

C. Arbitrarily angled faces

The in- and out- faces can be rotated through additional angles. If we use the MAD convention for specifying angles e_1 and e_2, effectively we must replace $\phi_i \to \phi_i + e_1$ and $\phi_f \to \phi_f - e_2$ in Eq. 14. The result then looks as follows.

$$(be^{-i\phi})_f = (be^{-i\phi})_i e^{-i\Delta\phi} + i\bar\rho(e^{-ie_2} - e^{ie_1}e^{-i\Delta\phi})$$

Setting $e_1 = -e_2 = \Delta\phi/2$ and using Eq. 15 again reproduces Eq. 13.

V. REPRESENTATION

The final complication arises when we insert the accelerator physicists' coordinate representation into these expressions. A particle's crossing the plane of a "face" is typically recorded using momentum and energy coordinates normalized by the fiducial. The position sector is recorded by transverse coordinates x_1, x_2, and a time offset, $c\Delta\tau$, while the momentum sector is represented by normalized transverse components[4] of $\vec{q} \equiv \vec{p}/\bar{p}$, and a total momentum offset, and $\delta \equiv |\vec{p}|/\bar{p} - 1$. To apply the bend algorithm, we need only write $\vec{v}$ in terms of these and then use the spinor $u = v_3 + iv_1$.

Starting from,

$$\vec\beta = \vec{v}/c = \vec{p}c/E = \vec{q} \cdot \bar{p}c/E \ ,$$

we substitute

$$E^2 = \vec{p}^2 c^2 + m^2 c^4 = (\bar{p}c)^2 (1+\delta)^2 + m^2 c^4 \ ,$$

to obtain

$$\vec\beta = \vec{q}/\sqrt{(1+\delta)^2 + (mc/\bar{p})^2} \ .$$

We can obtain v_1 and v_2 from this directly, but since q_3 is not recorded, we must get v_3 by using the following.

$$
\begin{aligned}
E^2 &= \vec{p}^2 c^2 + m^2 c^4 = \vec{p}_\perp^2 c^2 + p_3^2 c^2 + m^2 c^4 \\
\frac{E^2 - m^2 c^4}{\bar{p}^2 c^2} &= (\vec{q}_\perp)^2 + (q_3)^2 \\
\beta_3 &= q_3/\sqrt{\cdots} \\
&= \sqrt{\frac{(1+\delta)^2 - (\vec{q}_\perp)^2}{(1+\delta)^2 + (mc/\bar{p})^2}}
\end{aligned}
$$

Of course, there's even more fascinating material to wade through, but we're already up against the three page limit and have no more space (see edge of page below). A finished version will follow somewhere, sometime, but not here, and not now.

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[4] That is, the projections into the in- and out-faces.