Paraxial Expansion of a Static Magnetic Field in a Ring Accelerator*

Lee C. Teng
Argonne National Laboratory, 9700 So. Cass Ave., Argonne, Illinois 60439, U.S.A.

Abstract

In the paraxial approximation, trajectories of beam particles in a ring accelerator are computed expanded in powers of lateral displacements and slopes from a closed reference orbit. To do this, one needs first the expanded expressions of the magnetic field and potentials producing the particle motion. This is derived here in a most general form.

I. Introduction

In a storage ring or a ring accelerator the ideal closed orbit is generally a planar curve. All particles in the beam travel near to the closed orbit. Thus, for the study of the particle motion it is convenient to use the closed orbit as the reference axis and compute the particle trajectories in the paraxial approximation. The right-handed coordinates used are shown in Fig. 1 and are described below. Also shown in Fig. 1 is the local radius of

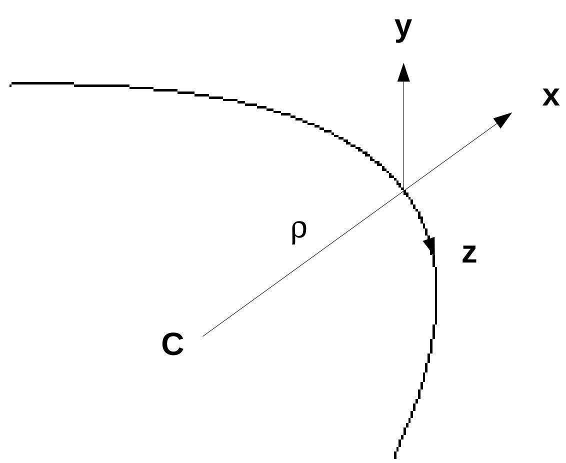

Figure 1
x: horizontal (in orbit plane) along the outward normal of the closed orbit, y: vertical, z: horizontal and along the forward tangent of the closed orbit.

curvature $\rho(z)$ and the center of curvature "C." The metric of these rotating coordinates is

$$ds^2 = dx^2 + dy^2 + (1+kx)^2 dz^2, \tag{1}$$

where

$$k = k(z) = 1/\rho(z) = \text{curvature}.$$

* Work supported by U.S. Department of Energy, Office of Basic Energy Sciences under Contract No. W-31-109-ENG-38.

II. Scalar Potential

To expand the paraxial particle trajectory in powers of x and y (and x' and y' with prime indicating d/dz) we need to first write the magnetic field (considered static) in an expanded form. In vacuum, the magnetic field can be expressed in terms of either a scalar potential ϕ or a vector potential A . It is simpler to start with the scalar potential. We shall write

$$\phi = \sum_{n=0}^{\infty}\sum_{m=0}^{n} a_{n,m}\frac{x^{n-m}}{(n-m)!}\frac{y^m}{m!} \equiv \sum_{n=0}^{\infty}\phi^{(n)}, \tag{2}$$

where the coefficients $a_{n,m} = a_{n,m}(z)$ are functions of z and the parenthesized superscript (n) indicates that the quantity is a homogeneous expression in x and y of total degree n.

The Laplace equations, $\nabla^2\phi = 0$, to be satisfied by ϕ is, in these coordinates,

$$\begin{aligned} -\frac{\partial^2\phi}{\partial y^2} \quad = \quad & \frac{1}{1+kx}\frac{\partial}{\partial x}\left[(1+kx)\frac{\partial\phi}{\partial x}\right] \\ & +\frac{1}{1+kx}\frac{\partial}{\partial z}\left(\frac{1}{1+kx}\frac{\partial\phi}{\partial z}\right). \end{aligned} \tag{3}$$

Straightforward but laborious expansions of $(1+kx)^{-p}$ factors and realignments of indices give the following recursion formula on index m

$$\begin{aligned} -a_{n+m+2,m+2} &= a_{n+m+2,m} \\ &+ \sum_{l=0}^{n}(-k)^l\frac{n!}{(n-l)!}[ka_{n-l+m+1,m} + (l+1)a''_{n-l+m,m} \\ &- \frac{(l+1)(l+2)}{2}(n-l)k'a'_{n-l+m-1,m}], \end{aligned} \tag{4}$$

where, as before, prime means d/dz. This is a rather messy and obscure formula. It is instructive to look at the first two recursions for $n = 0$ and 1,

<u>$n = 0$</u>

$$-a_{m+2,m+2} = a_{m+2,m} + ka_{m+1,m} + a''_{m,m}, \tag{5}$$

<u>$n = 1$</u>

$$\begin{aligned} -a_{m+3,m+2} &= a_{m+3,m} + ka_{m+2,m} \\ &+ a''_{m+1,m} - k'a'_{m,m} - k^2 a_{m+1,m} - 2ka''_{m,m}. \end{aligned} \tag{6}$$

These relations reveal clearly the hierarchy of the recursion. Because of the double step recursion in m, there are two sets of solutions. The field with odd m values starting with $m = 1$ is symmetric with respect to the orbit plane and is the normal design field. The field with even m values starting with $m = 0$ is anti-symmetric with respect to the orbit plane and is the skew field arising only from construction imperfections.

III. Magnetic Field Components

The components of the field B are given by $\mathsf{B} = \nabla\phi$, or

$$B_x = \frac{\partial\phi}{\partial x} = \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} a_{n,m}\frac{x^{n-m-1}}{(n-m-1)!}\frac{y^m}{m!} \equiv \sum_{n=1}^{\infty} B_x^{(n-1)},$$

$$B_y = \frac{\partial\phi}{\partial y} = \sum_{n=1}^{\infty}\sum_{m=1}^{n} a_{n,m}\frac{x^{n-m}}{(n-m)!}\frac{y^{m-1}}{(m-1)!} \equiv \sum_{n=1}^{\infty} B_y^{(n-1)}, \tag{7}$$

$$\begin{aligned} B_z &= \frac{1}{1+kx}\frac{\partial\phi}{\partial z} \\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{m=0}^{n-l}(-k)^l\frac{(n-m)!}{(n-m-l)!}a'_{n-l,m}\frac{x^{n-m}}{(n-m)!}\frac{y^m}{m!} \\ &\equiv \sum_{n=0}^{\infty} B_z^{(n)}. \end{aligned}$$

On the orbit plane $y = 0$, for the normal field ($m =$ odd) we have

$$\begin{aligned} B_y(y=0) &= \sum_{n=1}^{\infty} a_{n,1}\frac{x^{n-1}}{(n-1)!}, \\ B_x(y=0) &= B_z(y=0) = 0, \end{aligned} \tag{8}$$

and for the skew field ($m =$ even) we have

$$\begin{aligned} B_x(y=0) &= \sum_{n=1}^{\infty} a_{n,0}\frac{x^{n-1}}{(n-1)!}, \\ B_y(y=0) &= 0, \\ B_z(y=0) &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}(-k)^l\frac{n!}{(n-l)!}a'_{n-l,0}\frac{x^n}{n!}. \end{aligned} \tag{9}$$

The standard (American) multipole coefficients defined by

$$B_y + iB_x = B_0\sum(b_n + ia_n)(x+iy)^n, \quad i = \sqrt{-1}, \tag{10}$$

are related to the double indexed coefficient $a_{n,m}$ by

$$\begin{aligned} a_{n,0} &= \left(\frac{\partial^{n-1}B_x}{\partial x^{n-1}}\right)_{x=y=0} = B_0(n-1)!a_{n-1}, \\ a_{n,1} &= \left(\frac{\partial^{n-1}B_y}{\partial x^{n-1}}\right)_{x=y=0} = B_0(n-1)!b_{n-1}. \end{aligned} \tag{11}$$

IV. Vector Potential

In the vector potential A we choose the gauge $xA_x + yA_y = 0$ (the equivalent of $A_r = 0$ in the case of a linear reference orbit and cylindrical coordinates (r, θ, z). Then we can write

$$A_x = -yF, \quad A_y = xF, \quad A_z = \frac{G}{(1+kx)}, \tag{12}$$

where $F = F(x, y; z)$ and $G = G(x, y; z)$. The relation $\mathsf{B} = \nabla\times\mathsf{A}$ is, then

$$\begin{aligned} B_x &= \frac{1}{1+kx}\left(\frac{\partial G}{\partial y} - x\frac{\partial F}{\partial z}\right), \\ B_y &= \frac{1}{1+kx}\left(-\frac{\partial G}{\partial x} - y\frac{\partial F}{\partial z}\right), \\ B_z &= x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + 2F, \end{aligned} \tag{13}$$

or

$$\begin{aligned} x\frac{\partial G}{\partial x} + y\frac{\partial G}{\partial y} &= (1+kx)(yB_x - xB_y), \\ x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + 2F &= B_z. \end{aligned} \tag{14}$$

This shows that G or A_z is given by B_x and B_y, and that F, hence A_x and A_y, are given by B_z. Applying Euler's theorem for homogenous forms to give $F^{(p)}$ and $G^{(p)}$ in terms of $B_x^{(p)}$, $B_y^{(p)}$ and $B_z^{(p)}$, then substituting from Eq. (7) we get

$$\begin{aligned} F &= \sum_{n=0}^{\infty}\frac{1}{n+2}B_z^{(n)} \\ &= \frac{1}{2}a'_{0,0} + \sum_{n=1}^{\infty}\sum_{m=0}^{n}\frac{a'_{n,m}}{n+2}\frac{x^{n-m}}{(n-m)!}\frac{y^m}{m!}, \end{aligned} \tag{15}$$

and

$$\begin{aligned} G &= \sum_{n=1}^{\infty}\left(\frac{1}{n} + \frac{kx}{n+1}\right)\left(yB_x^{(n-1)} - xB_y^{(n-1)}\right) \\ &= \sum_{n=1}^{\infty}\left(\frac{1}{n} + \frac{kx}{n+1}\right)\times \\ &\quad \sum_{m=0}^{n} a_{n,m}\left[(n-m)\frac{y}{x} - m\frac{x}{y}\right]\frac{x^{n-m}}{(n-m)!}\frac{y^m}{m!}. \end{aligned} \tag{16}$$

Eqs. (12), (14), and (15) give the necessary expanded forms of the vector potential components for insertion into either the Lagrangian or the Hamiltonian function of the particle motion.

V. Lagrangian Formulation

To proceed we could employ either the Lagrangian or the Hamiltonian formulation. Here we present the simpler and more symmetric Lagrangian formulation. The orbit Lagrangian is

$$\begin{aligned} \mathsf{L}(x, x', y, y'; z) &= \left[(1+kx)^2 + x'^2 + y'^2\right]^{1/2} \\ &\quad + \frac{e}{p}\left[x'A_x + y'A_y + (1+kx)A_z\right] \\ &\equiv K(\text{kinematic term}) + D(\text{dynamic term}), \end{aligned} \tag{17}$$

where e and p are the charge and the momentum of the particle. The expanded forms of K and D are

$$K = (1+kx)\left[1 + \frac{x'^2 + y'^2}{(1+kx)^2}\right]^{1/2}$$

$$
\begin{aligned}
&= 1 + kx + \frac{1}{2}\left[\sum_{m=0}^{\infty}(-1)^m (kx)^m\right]\left(x'^2 + y'^2\right) \\
&\quad -\frac{1}{8}\left[\sum_{m=0}^{\infty}(-1)^m \frac{(m+1)(m+2)}{2}(kx)^m\right]\left(x'^2 + y'^2\right)^2 \\
&\quad + \cdots \qquad (18) \\
D &= \frac{1}{B\rho}\left[(y'x - x'y)F + G\right] \\
&= \frac{1}{B\rho}(y'x - x'y)\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{a'_{n,m}}{n+2}\frac{x^{n-m}}{(n-m)!}\frac{y^m}{m!} \\
&\quad + \frac{1}{B\rho}\sum_{n=1}^{\infty}\left(\frac{1}{n} + \frac{kx}{n+1}\right) \\
&\quad \times \sum_{m=0}^{n} a_{n,m}\left[(n-m)\frac{y}{x} - m\frac{x}{y}\right]\frac{x^{n-m}}{(n-m)!}\frac{y^m}{m!}, \qquad (19)
\end{aligned}
$$

where

$$B\rho \equiv \frac{p}{e} = \text{rigidity of the particle.}$$

It is easy to show that to the first degree terms one gets the well-known linear equations. To get the second- and higher-order terms the procedure is equally straightforward but increases progressively in complexity.