NONLINEAR DEPENDENCE OF SYNCHROTRON RADIATION ON BEAM PARAMETERS

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Abstract

Synchrotron radiation has been traditionally treated as an effect which only depends on the linear beam dynamics. Electrons in advanced accelerators and storage rings, however, can lose several percent of their energy in one turn, especially when the ring incorporates synchrotron radiation sources or free electron lasers. In these machines nonlinear effects can become important, not only because of the high variation of the particle's energy around the ring, but also because of the necessity to have very good beam quality in wigglers, undulators, and free electron lasers. Since these instruments can have helical structure, a general reference frame with torsion is used and the Lorenz–Dirac radiation reaction of the charged particle is taken into account. We will utilize the Differential Algebra technique to compute nonlinear transfer maps of general optical elements. Applications include radiation damping in multipoles, its effect on closed orbit distortion in a storage ring, and nonlinear tune shifts due to various radiating devices. The software provided will also be useful in simulating Siberian snakes.

I. INTRODUCTION

If the energy of a charged particle is conserved, the equations of motion are Hamiltonian. The Hamiltonian structure of these equations implies that their flows are symplectic. The symplectic symmetry poses several constraints on the motion. Liouville's theorem is an especially famous consequence of this property. If energy is lost during the motion, as in the case of electrons in a storage ring, the equations of motion are not Hamiltonian and knowledge about symplectic flows cannot be applied. If one sets out to describe high energy electron motion, it is therefore best not to start with a Hamiltonian but directly with the equation of motion. It is, however, not sufficient to solve the Hamiltonian equations of motion without radiation and to simulate an effective loss of energy after each dipole, since the phase space dependence of the lost energy is essential even in the linear theory [1]. The general equation of motion for charged particle optics, especially when used to compute highly energetic electron motion, should therefore include the energy loss due to radiation and the reaction of the particle to is own electromagnetic field.

The behavior of a charged particle in the superposition of an external field and its own retarded field cannot be described without problems [2], [3], [4], the classical limit is discussed in [5].

After an appropriate average in the Lorentz–Dirac equation, we reformulate the equations in such a way that they have a suitable form for the conventional description of particle optics. We will keep the argumentation completely general and allow reference curves with torsion. Thus we obtain the equations of motion with path length along some space curve as the independent parameter. These are the general equations of motion for describing charged particle optics.

To solve these equations of motion, we implemented the method of Taylor maps, which are computed with the Differential Algebraic (DA) technique. In standard devices like dipoles, quadrupoles, and higher order multipoles the equation of motion can be integrated with an exponential operator, such that the full power and speed of the DA method is obtained even with radiation effects.

II. THE LORENTZ–DIRAC EQUATION

To analyze the influence of radiation effects on particle motion consistently, we have to consider the field produced by a moving charged particle. Then we can compute the particle motion under the influence of its own field and an external field. To derive this equation, one can proceed in the following steps:

1. Derive the small velocity limit.
2. Write the general and covariant form of the equations of motion which satisfies the low velocity limit.
3. Reformulate the differential equation into an integral equation.
4. Approximate the integral equation.
5. Transform the covariant equation into a specific inertial frame.

These steps can be extracted from the references mentioned. As a result one obtains the equation of motion

$$m\frac{d}{d\tau}u^\mu = e\sqrt{\frac{\mu_0}{4\pi}}F^{\mu\nu}u_\nu + \frac{e^2\mu_0}{6\pi c^3}u^\mu\frac{d}{d\tau}u_\nu\frac{d}{d\tau}u^\nu \qquad (1)$$

which in the lab frame is

$$\frac{d}{dt}\vec{p} = \vec{F} - \frac{e^2\mu_0\gamma^4}{6\pi c^3}[\vec{a}^2 + \frac{\gamma^2}{c^2}(\vec{a}\vec{v})^2]\vec{v}\,. \qquad (2)$$

with the external force $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$. In particle optics we require an equation of motion of the form $\vec{z}' = \vec{f}(\vec{z}, s)$ for phase space variables $\vec{z}$. It indeed turns out that such a formulation is possible. After introduction of

$$\frac{d}{dt}\vec{p} = m\gamma\vec{a} + m\frac{\gamma^3}{c^2}(\vec{v}\vec{a})\vec{v} \qquad (3)$$

and several algebraic manipulations one is led to

$$\begin{aligned}\delta &= \frac{e^2\mu_0\gamma^2 v}{6\pi m^2 c^3}\,, \qquad (4)\\ \frac{d}{dt}\vec{p} &= \vec{F}_\perp + \vec{n}\frac{\gamma^2}{2\delta}[\sqrt{1 + 4\frac{\delta}{\gamma^2}(\vec{n}\vec{F} - \delta(\vec{n}\times\vec{F})^2)} - 1]\,.\end{aligned}$$

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We can expand to the first power of $\delta|\vec{F}|$ and obtain

$$\frac{d}{dt}\vec{p} = \vec{F} - \vec{n}\delta[(\vec{n}\times\vec{F})^2 + \frac{1}{\gamma^2}(\vec{n}\vec{F})^2] \; . \tag{5}$$

This expansion is valid even for very high energy. For an electric field of $10^7\frac{\mathrm{V}}{\mathrm{m}}$, this electron energy would be around 2700GeV. For a magnetic field of 10T, this energy is about 150GeV.

III. THE CURVILINEAR COORDINATE SYSTEM

We want to introduce a coordinate system with which particle motion close to some reference curve $\vec{R}(s)$ parameterized by its path length s can be described well. For that purpose one defines s dependent unit vectors $\vec{e}_x$ and $\vec{e}_y$ in such a way that close to this curve space points $\vec{r}$ can be expressed as

$$\vec{r} = \vec{R}(s) + x\vec{e}_x(s) + y\vec{e}_y(s) \tag{6}$$

and the unit vectors $\vec{e}_s = \partial_s\vec{R}$, $\vec{e}_x$, and $\vec{e}_y$ build a right handed coordinate frame. The reference curve's curvature κ can have x and y components κ_x and κ_y. There are three reasonable possibilities with different forms for their s derivatives.

1. The horizontal system in which one chooses $\vec{e}_x$ always perpendicular to one fixed space direction $\vec{e}_1$:

$$\begin{aligned}\frac{d}{ds}\vec{r} &= (x' + \frac{y\kappa_x}{\tan(\angle\vec{e}_1\vec{e}_s)})\vec{e}_x \\ &+ (y' - \frac{x\kappa_x}{\tan(\angle\vec{e}_1\vec{e}_s)})\vec{e}_y + (1 + x\kappa_x + y\kappa_y)\vec{e}_s \; .\end{aligned} \tag{7}$$

2. The Frenet coordinate system in which one chooses the $\vec{e}_x$ direction always parallel to the curvature vector, which rotates around the reference curve with the torsion T:

$$\frac{d}{ds}\vec{r} = (x' - yT)\vec{e}_x + (y' + xT)\vec{e}_y + (1 + x\kappa)\vec{e}_s \; . \tag{8}$$

3. The curvilinear system in which the effect of the torsion is compensated by a reversed rotation [6]. In this case the orientation of the coordinate system at s depends on the history of the torsion between 0 and s and is therefore not defined locally. This disadvantage can be compensated by the simplicity of

$$\frac{d}{ds}\vec{r} = x'\vec{e}_x + y'\vec{e}_y + (1 + x\kappa_x + y\kappa_y)\vec{e}_s \; . \tag{9}$$

IV. THE LORENTZ–DIRAC EQUATION IN CURVILINEAR COORDINATES

In this coordinate system we therefore have

$$\begin{aligned} h &= 1 + x\kappa_x + y\kappa_y \; , \\ \frac{d}{ds}\vec{r} &= x'\vec{e}_x + y'\vec{e}_y + h\vec{e}_s \; , \\ \vec{p} &= p_x\vec{e}_x + p_y\vec{e}_y + p_s\vec{e}_s \; , \\ \frac{d}{ds}\vec{p} &= (p_x' - p_s\kappa_x)\vec{e}_x + (p_y' - p_s\kappa_y)\vec{e}_y \\ &+ (p_s' + p_x\kappa_x + p_y\kappa_y)\vec{e}_s \; . \end{aligned} \tag{10}$$

We use standard particle optical phase space coordinates

$$a = \frac{p_x}{p_0} \; , \; \; b = \frac{p_y}{p_0} \; , \; \; \tau = (t_0 - t)\frac{K_0}{p_0} \; , \; \; \delta = \frac{K}{K_0} \tag{11}$$

where subscripts 0 refer to a reference particle, which in general does not have to follow the reference curve $\vec{R}(s)$ however. By defining

$$\begin{aligned} \mathcal{E} &= E_x a + E_y b + E_s\frac{p_s}{p_0} \; , \\ \mathcal{B} &= B_x a + B_y b + B_s\frac{p_s}{p_0} \; , \end{aligned} \tag{12}$$

the general equations of motion which are obtained with $ds/dt = v/h \cdot p_s/p$ simplify to

$$\begin{aligned} x' &= ha\frac{p_0}{p_s} \; , \; \; y' = hb\frac{p_0}{p_s} \; , \\ \tau' &= (\frac{h_0p_0}{v_0p_{s0}} - \frac{hp}{vp_s})\frac{K_0}{p_0} \; , \\ a' &= \frac{qh}{p_sp_0}\{m\gamma E_x + p_0(bB_s - \frac{p_s}{p_0}B_y)\} + \frac{p_s}{p_0}\kappa_x - \zeta a \; , \\ b' &= \frac{qh}{p_sp_0}\{m\gamma E_y - p_0(aB_s - \frac{p_s}{p_0}B_x)\} + \frac{p_s}{p_0}\kappa_y - \zeta b \; , \\ \delta' &= \frac{qhp_0}{p_sK_0}\mathcal{E} - \xi \; . \end{aligned} \tag{13}$$

where the damping terms are

$$\begin{aligned} \zeta &= \frac{\mu_0 q^4 v}{6\pi m^2 c^3}(\frac{p_0}{p})^3\{\mathcal{E}^2 + \gamma^2(\\ & [bE_s - \frac{p_s}{p_0}E_y - \frac{p_0}{m\gamma}\{B_x(\frac{p}{p_0})^2 - a\mathcal{B}\}]^2 \\ &+ [\frac{p_s}{p_0}E_x - aE_s - \frac{p_0}{m\gamma}\{B_y(\frac{p}{p_0})^2 - b\mathcal{B}\}]^2 \\ &+ [aE_y - bE_x - \frac{p_0}{m\gamma}\{B_s(\frac{p}{p_0})^2 - \frac{p_s}{p_0}\mathcal{B}\}]^2)\} \; , \\ \xi &= \zeta\frac{p^2}{m\gamma K_0}. \end{aligned} \tag{14}$$

It is worth mentioning that the right hand side can be chosen origin preserving and s independent in standard devices like dipoles, quadrupoles, and multipoles. The transfer map can therefore be computed with the exponential operator $\exp(L_{\vec{f}})$ [7] and the full speed and power of the DA technique can be used even with radiation effects.

V. THE IMPLEMENTATION

Routines were written which integrate the equation of motion with the 8^{th} order Runge Kutta integrator of the DA program COSY INFINITY[8]. By integrating with DA techniques, one obtains the phase space curve of the reference particle $\vec{z}_0(s)$ simultaneously with the Taylor map $\vec{M}$ to arbitrary order such that $\vec{z}(s) = \vec{z}_0(s) + \vec{M}[\vec{z}(0) - \vec{z}_0(0)]$ for particles starting in phase space at $\vec{z}(0)$ close to the reference particle at $\vec{z}_0(0)$.

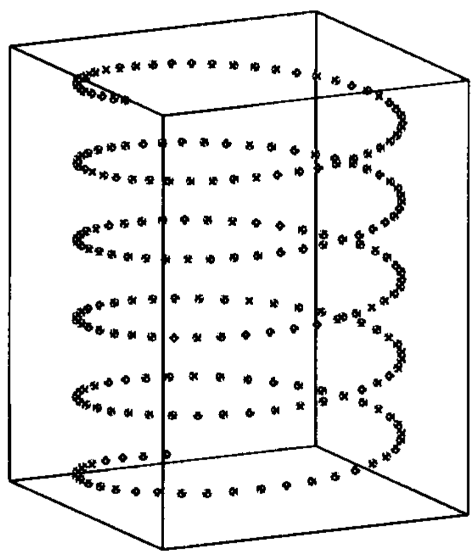

Figure 1. No radiative energy loss: $|\partial_{\vec{z}} \vec{M}^T| = 1$

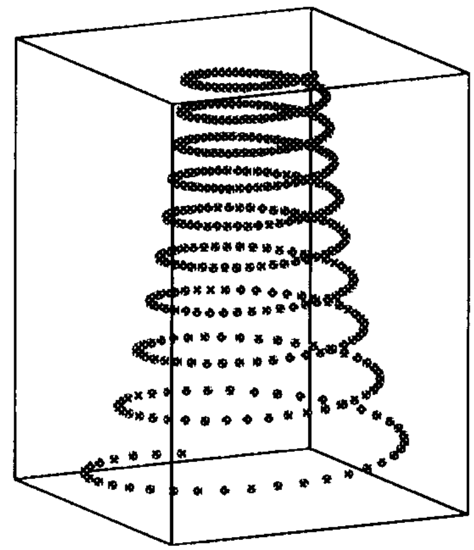

Figure 2. With radiative energy loss: $|\partial_{\vec{z}} \vec{M}^T| = 0.023$

The figures (1) and (2) show an electron's path through a constant magnetic field with and without synchrotron radiation. The second order transfer maps are displayed in the adjacent tables. The Jacobians of the linear transfer maps are 1 for the energy conserving calculations and 0.023 for the calculations with radiation. This test manifests what is known as phase space damping in electron synchrotrons.

Our main purpose in using the constructed software will be the analysis of nonlinear effects in helical structures in proton machines (Siberian snakes) and electron machines (wigglers). Also the possibilities of influencing damping distributions with multipole wigglers should be analyzed.

x	a	y	b	τ	δ	powers
37.9	-.253	52.7	.758	0	0	000000
1	0	0	0	0	0	100000
-893	-56.4	2876	-17.8	-3704	0	010000
0	0	1	0	0	0	001000
-65.9	-.948	47.4	-.317	0	0	000100
0	0	0	0	1	0	000010
1176	70.1	-3512	23.4	2963	1	000001
-1E5	403	-2E4	-2176	-2E4	0	020000
-3512	23.4	-1176	-70.1	0	0	010100
.3E6	-1183	.6E5	5382	.3E5	0	010001
-587	-35.1	1756	-11.7	-2315	0	000200
4390	-29.3	1469	87.7	0	0	000101
-2E5	829	-4E4	-3333	-1E4	0	000002

x	a	y	b	τ	δ	powers
-6.59	-.252	52.4	-.171	0	-.6	000000
1	0	0	0	0	0	100000
-3301	45.1	-2189	-65.7	-3704	-1	010000
0	0	1	0	0	0	001000
-65.5	.214	-8.24	-.316	0	0	000100
0	0	0	0	1	0	000010
3066	-41.9	2077	60.9	2963	1.0	000001
3E5	8774	-4E5	5646	-1E4	-.5	020000
2818	81.9	-4116	56.8	0	0	010100
-5E5	-2E4	8E5	-1E4	3E4	1.1	010001
-2058	28.4	-1409	-40.9	-2315	-.7	000200
-2596	-76.2	3833	-52.4	0	0	000101
2E5	7545	-4E5	4859	-1E4	-.7	000002

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