LINEAR ORBIT PARAMETERS FOR THE EXACT EQUATIONS OF MOTION*

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Abstract

This paper defines the beta function and other linear orbit parameters using the exact equations of motion. The β, α and ψ functions are redefined using the exact equations. Expressions are found for the transfer matrix and the emittance. The differential equations for $\eta = x/\beta^{1/2}$ is found. New relationships between α, β, ψ and ν are derived.

I. INTRODUCTION

This paper defines the beta function and the other linear orbit parameters using the exact equations of motion. The usual treatment [1] of the linear orbit parameters is based on the approximate equation of motion

$$\frac{d^2x}{ds^2} + K(s)x = 0 \tag{1}$$

Approximations are made in obtaining Eq. (1) which are usually valid for large accelerators.

The exact linearized equations of motion can be written as

$$\begin{aligned} \frac{dx}{ds} &= A_{11}x + A_{12}p_x \\ \frac{dp_x}{ds} &= A_{21}x + A_{22}p_x \end{aligned} \tag{2}$$

x and p_x are the canonical coordinates in a curvilinear coordinate system based on a reference orbit and the $A_{ij}(s)$ are periodic in s with period L. The approximate Eq. (1) assumes that $A_{11} = A_{22} = 0$, $A_{12} = 1$ and $A_{21} = -K(s)$. The exact values of the A_{ij} are given in [2].

A treatment of the linear orbit parameters based on the exact equations, Eqs. (2), rather than the approximate Eq. (1) may be desirable in the following situations:

1. Symplectic long term tracking using a procedure where the magnets are replaced by a sequence of point magnets and drift spaces. For the tracking to be symplectic, one has to use the solutions of the exact equations of motion. The linearized equations of motion then have the form of Eq. (2).
2. Small accelerators where the approximations made in deriving Eq. (1) may not be valid.

Many of the results found using the approximate equations carry over for the exact equations. A few of the changed results are the following:

$$\begin{aligned} \alpha &= \frac{1}{A_{12}}\left(-\frac{1}{2}\frac{d\beta}{ds} + A_{11}\beta\right) \\ \psi &= \int_0^s A_{12}\frac{ds}{\beta} \end{aligned} \tag{3}$$

$$\nu = \frac{1}{2\pi}\int_0^C ds\frac{A_{12}}{\beta}$$

where C is the circumference of the accelerator.

II. EIGENFUNCTIONS OF THE EXACT LINEAR EQUATIONS OF MOTION AND THE LINEAR ORBIT PARAMETERS

The problem now is, given the exact linear equations of motion, Eq. (2), how does one define the linear orbit parameters $\beta, \alpha, \gamma, \psi, \nu$ and the emittance ϵ, and what are the relationships that hold between them. To do this, one has to repeat the well known treatment of the linear orbit parameters, and see where the definitions and relationships change for the exact equations. The treatment given below is believed to reduce the amount of algebraic manipulation required, and makes few assumptions about the A_{ij} coefficients in the linear equations.

For the x motion, the linear equations are written as

$$\begin{aligned} \frac{dx}{ds} &= A_{11}x + A_{12}p_x \\ \frac{dp_x}{ds} &= A_{21}p_x + A_{22}x \end{aligned} \tag{4}$$

The transfer matrix $M(s, s_0)$ obeys

$$\begin{aligned} x &= M(s, s_0)x_0 \\ x &= \begin{pmatrix} x \\ p_x \end{pmatrix} \\ \frac{d}{ds}M &= AM \end{aligned} \tag{5}$$

One may note that the symbol x is used in 2 different ways. The meaning of x should be clear from the context. The matrix M is symplectic as the equations of motions are derived from a hamiltonian. [1,3] Thus

$$\begin{aligned} M\overline{M} &= I \\ \overline{M} &= \tilde{S}\tilde{M}\, S \\ S &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{6}$$

$\tilde{S}$ is the transverse of S. Also $|M| = 1$; $|M|$ is the determinant of M.

The one period transfer matrix is defined by

$$\hat{M}(s) = M(s+L, s) \tag{7}$$

where L is the period of the A_{ij} in Eq. (4). One can show that $\hat{M}(s)$ and $\hat{M}(s_0)$ are related by

$$\hat{M}(s) = M(s, s_0)\hat{M}(s_0)M(s_0, s) \tag{8}$$

*Work performed under the auspices of the U.S. Department of Energy.

The eigenfunctions and eigenvalues of $\hat{M}(s)$ are defined by

$$\begin{aligned} \hat{M}(s)x &= \lambda x, \\ |\hat{M} - \lambda I| &= 0, \\ \lambda^2 - (m_{11} + m_{22})\lambda + 1 &= 0 \end{aligned} \tag{9}$$

where m_{ij} are the elements of $\hat{M}$, and using $|\hat{M}| = 1$.

Eqs. (9) shows that the two eigenvalues λ_1, λ_2 obey

$$\lambda_1 \lambda_2 = 1, \tag{10}$$

and for stable motion, $|\lambda| = 1$ and $\lambda_2 = \lambda_1^*$, and we can write

$$\lambda_1 = \exp(i\mu) \tag{11}$$

Given the eigenfunction at s_0, $x_1(s_0)$ one can find the eigenfunction or any other point s using

$$x_1(s) = M(s, s_0) x_1(s_0), \tag{12}$$

and $x_1(s)$ has the same eigenvalue λ_1. This follows from Eq. (9), using Eq. (8) to relate $\hat{M}(s)$ and $\hat{M}(s_0)$. Also $x_1(s)$ obeys the linear equations of motion,

$$\frac{d}{ds} x_1 = A x_1, \tag{13}$$

which follows from Eq. (12) and Eq. (5). One can show that

$$x_1(s)/\lambda_1^{s/L} = f_1(s), \tag{14}$$

where $f_1(s + L) = f_1(s)$. This follows from

$$\begin{aligned} f_1(s+L) &= x_1(s+L)/\lambda_1^{s/L+1} \\ &= \hat{M}(s) x_1(s)/\lambda_1^{s/L+1} = x_1(s)/\lambda_1^{s/L} \end{aligned}$$

Thus, one can write

$$\begin{aligned} x_1(s) &= \exp(i\mu s/L)\, f_1(s) \\ f_1(s+L) &= f_1(s) \end{aligned} \tag{15}$$

Eq. (15) can be rewritten as

$$\begin{aligned} x_1(s) &= \beta(s)^{1/2} \exp(i\psi) \\ \psi(s) &= \mu s/L + g_1(s) \\ g_1(s+L) &= g_1(s), \quad \beta(s+L) = \beta(s) \end{aligned} \tag{16}$$

Eq. (16) defines the beta functions, $\beta(s)$, except for a normalization multiplyer, for the eigenfunction $x_1(s)$. The normalization multiplyer will be defined below. It will be shown first that ψ and β are related. To find this relation, one uses the Lagrange invariant [1]

$$W = \tilde{x}_2\, S x_1 \tag{17}$$

where x_1, x_2 are two solutions of the equations of motion. Eq. (17) corresponds to the Wronskian in the treatment of the approximate equations of motion. For x_1 and x_2, we use the two eigenfunctions x_1 and $x_2 = x_1^*$.

$$x_1 = \begin{pmatrix} x_1 \\ p_{x1} \end{pmatrix} \tag{18}$$

For x_1 one uses Eq. (16) and for p_{x1} one finds from the equations of motion

$$p_{x1} = \frac{1}{A_{12}} \left(\frac{dx_1}{ds} - A_{11} x \right) \tag{19}$$

$$\begin{aligned} W &= x_2 p_{x1} - p_{x2} x_1 \\ W &= \left[x_2 \frac{dx_1}{ds} - \frac{x_1 dx_2}{ds} \right] \frac{1}{A_{12}} \\ W &= \frac{2i}{A_{12}} \beta \frac{d\psi}{ds} \end{aligned} \tag{20}$$

The beta function β is normalized by normalizing the eigenfunctions so that

$$W = \tilde{x}_1^*\, S x_1 = 2i \tag{21}$$

which gives

$$\frac{d\psi}{ds} = \frac{A_{12}}{\beta} \tag{22}$$

Eq. (22) replaces the familiar result $d\psi/ds = 1/\beta$ which is obtained when $A_{12} = 1$. From Eq. (22) one can find a result for the tune. Using $2\pi\nu = \psi(C) - \psi(0)$ where C is the circumference of the accelerator, one finds

$$\nu = \frac{1}{2\pi} \int_0^C ds \frac{A_{12}}{\beta} \tag{23}$$

From Eq. (19) we now find for p_{x1},

$$p_{x1} = \frac{1}{\beta^{1/2}} (i - \alpha) \exp(i\psi) \tag{24}$$

$$\alpha = \frac{1}{A_{12}} \left(-\frac{1}{2} \frac{d\beta}{ds} + A_{11} \beta \right) \tag{25}$$

Eq. (25) provides the new definition for the α parameter, which replaces the familiar result $\alpha = -\frac{1}{2} d\beta/ds$. At this point the definition of α may seem arbitrary. It will be seen to be the convenient definition of α when the emittance and transfer matrix are considered.

The eigenfunctions can now be written as, using Eq. (16) and Eq. (25),

$$\begin{aligned} x_1 &= \begin{bmatrix} \beta^{\frac{1}{2}} \\ \beta^{-\frac{1}{2}}(-\alpha + i) \end{bmatrix} \exp(i\psi) \\ x_2 &= x_1^* \end{aligned} \tag{26}$$

For the results for the emittance and transfer matrix, see [2].

III. DIFFERENTIAL EQUATIONS FOR THE LINEAR ORBIT PARAMETERS

This section finds differential equations for β, and η

A. Second Order Differential Equation for x

From the first order differential equation for x, p_x, Eq. (4), one can eliminate p_x to find a second order equation for x. See [2] for details

$$\frac{d}{ds} \left(\frac{1}{A_{12}} \frac{dx}{ds} \right) + x \left(-A_{21} - \frac{d}{ds} \left(\frac{A_{11}}{A_{12}} \right) - \frac{A_{11}^2}{A_{12}} \right) = 0 \tag{27}$$

It has been assumed that $A_{11} = -A_{22}$.

B. Differential Equation for β

To find a differential equation for β, into Eq. (27) for x put the eigenfunction

$$\begin{aligned} x &= b\ \exp(i\psi) \\ b &= \beta^{1/2} \end{aligned} \tag{28}$$

We find then, see [2] for details,

$$\frac{d}{ds}\left(\frac{1}{A_{12}}\frac{db}{ds}\right) - \frac{A_{12}}{b^3} + b\left(-A_{21} - \frac{d}{ds}\left(\frac{A_{11}}{A_{12}}\right) - \frac{A_{11}^2}{A_{12}}\right) = 0 \tag{29}$$

Eq. (29) is a second order differential equation for $b = \beta^{1/2}$. It can be compared to the result found when $A_{12} = 1$ and $A_{11} = 0$,

$$\frac{d^2 b}{ds^2} - \frac{A_{12}}{b^3} = 0 \tag{30}$$

C. Differential Equation for η

η and x are related by

$$x = b\ \eta, \qquad b = \beta^{1/2} \tag{31}$$

In the differential equation for η the independent variable is ψ or θ which are related to s by

$$\begin{aligned} d\psi &= A_{12}\frac{ds}{\beta} \\ d\theta &= A_{12}\frac{ds}{\nu\beta} \end{aligned} \tag{32}$$

We find dx/ds and $d(A_{12}^{-1}dx/ds)/ds$ which are then substituted into Eq. (27) to get the equation for η, using Eq. (29) to eliminate derivatives of b. This gives, see [2] for details,

$$\frac{d^2\eta}{d\theta^2} + \nu^2\eta = 0 \tag{33}$$

The differential equation for η is unchanged.

IV. PERTURBATION THEORY USING THE DIFFERENTIAL EQUATION FOR η

The equation for η, Eq. (33) is often used as a starting point in finding the effects of a perturbing field. The particle coordinates are measured relative to a reference orbit which is the particle motion in a known magnetic field with components B_i. The exact equations of motion can then be written as

$$\frac{dx_i}{ds} = \sum_j A_{ij}x_j + f_i \qquad i = 1,4,\ j = 1,4 \tag{34}$$

where the f_i includes all the terms not included in $\sum A_{ij}x_j$. These include terms due to fields not included in the reference field B_i, which may be referred as ΔB_i, and nonlinear terms due to the terms in the exact equations of motion that do not depend on B_i.

One can see from the exact equations of motion, that the contributions to f_i which depend explicitly on ΔB_i, when $\Delta B_s = 0$, are given by

$$\begin{aligned} f_2 &= \frac{1}{B\rho}(1 + x/\rho)\Delta B_y \\ f_4 &= -\frac{1}{B\rho}(1 + x/\rho)\Delta B_x \end{aligned} \tag{35}$$

Repeating the above derivation of Eq. (33) for η, including the f_i terms, one finds the η equation for the x-motion

$$\begin{aligned} \frac{d^2\eta}{d\theta^2} + \nu_x^2\eta &= \frac{\nu_x^2\beta_x^{3/2}}{A_{12}} f_x \\ f_x &= f_2 + \frac{A_{11}^2}{A_{12}} f_1 + \frac{d}{ds}\left(\frac{f_1}{A_{12}}\right) \\ d\theta &= A_{12}\frac{ds}{\nu_x\beta_x} \end{aligned} \tag{36}$$

A similar equation can be found for the y motion,

$$\begin{aligned} \frac{d^2\eta}{d\theta^2} + \nu_y^2\eta &= \frac{\nu_y^2\beta_y^{3/2}}{A_{34}} f_y \\ f_y &= f_4 + \frac{A_{33}^2}{A_{34}} f_3 + \frac{d}{ds}\left(\frac{f_3}{A_{34}}\right) \end{aligned} \tag{37}$$

For the case of a gradient perturbation

$$\Delta B_y = -Gx \tag{38}$$

one can use Eq. (36) to find the change in ν_x, $\Delta\nu_x$. One finds

$$\Delta\nu_x = \frac{1}{4\pi}\int ds\beta_x\frac{G}{B\rho} \tag{39}$$

This well known result for $\Delta\nu_x$ is not changed by using the exact linear equations.

References

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[3] F.T. Cole, Notes on Accelerator Theory, MURA report, TN-259 (1961).