SUM BETATRON RESONANCES UNDER LINEAR COUPLING OF OSCILLATIONS

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Abstract

The joint effect of an arbitrary sum resonance and a linear coupling resonance $Q_y - Q_x = 0$ on stability of betatron oscillations in a circular accelerator is studied. The presence of linear coupling is shown to result in splitting of the cluster of sum resonance straight-lines into a family of hyperbolic curves. The analytic results obtained are verified by numerical simulations.

I. INTRODUCTION

At present, the design employed of SC dipoles for high energy accelerators does not ensure the small enough values of the skew quadrupole field errors, Refs.[1],[2]. This gradient brings about the major contribution to linear coupling between horizontal (x) and vertical (y) betatron oscillations.

The effect of this coupling on the motion has been studied in many papers (refer, say, to Refs.[3]–[8]) of which we would distinguish quite a rigorous and consistent Ref.[8]. These papers treat the problem in terms of normal oscillation modes and the relevant tunes $Q_{1,2}$, the latter being quite distinct from the unperturbed betatron tunes $Q_{x,y}$. However, to analyze the joint effect of linear coupling and magnetic nonlinearities, it might be more convenient to employ x, y-modes of the unperturbed oscillations.

In frames of the first-order perturbation approach, linear coupling shows itself up as an excitation of sum and difference resonances $Q_y \pm Q_x = k$. Being treated isolately from the other resonances, the linear difference resonance is, by itself, not dangerous for the motion of beam with equal betatron x- and y-emittances. The total energy of 2-D oscillations being kept intact, this resonance gives rise to energy exchange between x- and y-directions. Nonetheless, such a resonance can result in an unstable motion in the presence of an additional sum resonance (not necessarily driven by the skew quadrupole) which is far enough from the working point not to inflict any danger, provided the `switched-off' linear difference resonance.

The common vision that the loss of stability occurs only on the condition $\vec{n}\vec{Q} = k$ being satisfied seems to be not quite the case. It does hold true for the isolated sum resonance, given there exists only one such a resonant straight line in plane $\{Q_x, Q_y\}$ near the working point. By a simple example of joint action of an arbitrary n-th order sum resonance and a linear difference resonance, it would be shown here that there exists a family of $(n+1)$ hyperbolic curves in the betatron tune plane at which the loss of stability is possible.

II. ANALYTIC RESULTS

A. Betatron Oscillations of the Reference Particle

Components $h_{x,y}$ of the magnetic field imperfections are expressed in terms of the longitudinal vector potential A_s :

$$h_x = -\partial A_s/\partial y, \quad h_y = \partial A_s/\partial x,$$

$$A_s(x,y,s) = \sum_{n=0}^{\infty} \frac{\Delta H_{y,n} + i\Delta H_{x,n}}{2r^n(n+1)}(x+iy)^{n+1} + \text{c.c.}$$

where $\Delta H_{y,n}(s), \Delta H_{x,n}(s)$ are additions to the field introduced by the n-th order normal and skew nonlinearities, respectively, taken at $(x = r, y = 0)$; r is a reference radius.

Up to the first order in perturbation, equations of betatron motion of the on-momentum particle in such a field acquire the canonical form, Ref.[9]:

$$\frac{dI_\zeta}{d\theta} = -\frac{\partial\langle D\rangle}{\partial\eta_\zeta}, \qquad \frac{d\eta_\zeta}{d\theta} = \frac{\partial\langle D\rangle}{\partial I_\zeta}, \tag{1}$$

$$D = \frac{2\beta_{max}R_0}{r^2HR}A_s(\zeta,\theta),$$

$$\zeta = r\sqrt{\beta_\zeta(\theta)/\beta_{max}}\sqrt{I_\zeta}\cos(\mu_\zeta(\theta)+\eta_\zeta)$$

where ζ is an equation symbol for either x or y; R_0 and R are the average and curvature radii, respectively, of the reference orbit in field H; θ is a generalized azimuth which may be expressed by the longitudinal coordinate s as $\theta = s/R_0$; $\beta_\zeta(\theta)$ is beta-function and $\mu_\zeta = Q_\zeta\theta + \chi_\zeta$ is unperturbed phase with a periodic part $\chi_\zeta(\theta)$. Thus, $\sqrt{I_\zeta}$ is the ζ-oscillation amplitude normalized to r and taken at azimuth where $\beta_\zeta = \beta_{max}$. The operator $\langle\ldots\rangle$ denoting the averaging over θ removes fast harmonics.

By taking into account the periodic dependence of D on $\vec{\mu} = (\mu_x, \mu_y)$ and azimuth θ, one can put down

$$D = \sum_{k=-\infty}^{\infty}\sum_{\vec{n}} D_{\vec{n},k}(I_x, I_y)\exp\{i(\vec{n}\vec{Q}-k)\theta + i\vec{n}\vec{\eta}\}$$

$$D_{\vec{n},k} = \frac{1}{(2\pi)^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} D\{r\sqrt{\frac{\beta_\zeta(\theta)}{\beta_{max}}}\sqrt{I_\zeta}\cos\alpha_\zeta, \theta\} \times \exp\{i[k\theta + \vec{n}\vec{\chi}(\theta) - \vec{n}\vec{\alpha}]\}\, d\alpha_x d\alpha_y d\theta$$

where $\vec{n} = (n_x, n_y), \quad n_{x,y}, k$ are integers.

The resonant harmonics enter $\langle D\rangle$ as complex conjugated pairs, $\{\vec{n}, k\}$ and $\{-\vec{n}, -k\}$, which are responsible for excitation of $\vec{n}\vec{Q} = k$ resonance. The infinite increase of the total energy $I = I_x + I_y$ is possible under the impact of the isolated sum resonance $\vec{n}$ with $n_x \cdot n_y > 0$. Due to this reason, the 1-D resonances $\vec{n} = (n_x, 0)$ and $\vec{n} = (0, n_y)$ should rather be treated as sum ones. Simplify expression for amplitude $D_{\vec{n},k}$ of the resonant harmonic by retaining the contribution only from the nonlinearity of minimal power allowed for the given order of resonance. Then

$$D_{\vec{n},k} = P_{\vec{n},k}\sqrt{I_x^{|n_x|}I_y^{|n_y|}},$$

$$P_{\vec{n},k} = \frac{\beta_{max} R_0}{\pi r n} \binom{n}{|n_y|} \int_0^{2\pi} W_{\vec{n}} \sqrt{\frac{\beta_x^{|n_x|} \beta_y^{|n_y|}}{(4\beta_{max})^n}} e^{i(k\theta + \vec{n}\vec{\chi})} d\theta$$

$$W_{\vec{n}} = \left\{ \frac{\Delta H_{y,n-1}}{RH} \cos \frac{\pi}{2}|n_y| - \frac{\Delta H_{x,n-1}}{RH} \sin \frac{\pi}{2}|n_y| \right\}$$

where $n = |n_x| + |n_y|$ is the order of a resonance.

Let the cross-point $\vec{Q}_*$ of the n-th order sum resonance with the line of difference resonance $\vec{m}\vec{Q} = p$, $\vec{m} = (-1, 1)$ be referred to as the n-th order cluster. No more than $(n+1)$ sum resonances of n-th order can cross such a node: $\vec{n} = (n-j, j)$ where $j = 0, \ldots, n$. All these resonances can be driven by the same $(n-1)$-th power field nonlinearity.

The joint effect of a sum resonance $\vec{n}$ and a difference resonance $\vec{m}$ near $\vec{Q}_*$ is described in terms of variables $\{\vec{I}, \vec{\eta}\}$ through the canonical Eqs.1 with Hamiltonian:

$$\begin{aligned} \langle D \rangle &= \{P_{\vec{m},p} e^{i\vec{m}\vec{w}} + \text{c.c.}\} \sqrt{I_x I_y} \\ &+ \{P_{\vec{n},k} e^{i\vec{n}\vec{w}} + \text{c.c.}\} \sqrt{I_x^{n_x} I_y^{n_y}} \end{aligned} \quad (2)$$

where $\vec{w} = \vec{\delta}\theta + \vec{\eta}$; $\vec{\delta} = (\vec{Q} - \vec{Q}_*)$ is the working point detuning from the cluster; $n_{x,y} \geq 0$.

B. Effect of Isolated Difference Resonance $\vec{m}$

Study of motion in the vicinity of the isolated difference resonance $\vec{m}$ shows that any particle has its $I_{x,y}$ varying harmonically with a frequency $2\omega = 2\sqrt{|P_{\vec{m},p}|^2 + (\vec{m}\vec{\delta}/2)^2}$. Frequency ω does not depend on oscillation phase and amplitude, which allows one to transfer to new variables $I_{1,2}$ and $\eta_{1,2}$, the latter being the integrals of motion,

$$\begin{pmatrix} \sqrt{I_x} e^{i\eta_x} \\ \sqrt{I_y} e^{i\eta_y} \end{pmatrix} = T(\psi)\, M(\alpha)\, T(-\omega\theta) \begin{pmatrix} \sqrt{I_1} e^{i\eta_1} \\ \sqrt{I_2} e^{i\eta_2} \end{pmatrix} \quad (3)$$

$$T(\psi) = \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix}, \quad M(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

$$2\psi = \vec{m}\vec{\delta}\theta + \arg P_{\vec{m},p}, \quad \tan\alpha = \frac{|P_{\vec{m},p}|}{\omega + \vec{m}\vec{\delta}/2}, \ \ 0 < \alpha < \pi/2.$$

Transformation by Eq.3 does not change expression for the total energy $I = I_x + I_y = I_1 + I_2$.

C. Effect of Difference Resonance and Sum Resonance

Whenever simultaneous effect of a sum, $\vec{n}$, and a difference, $\vec{m}$, resonances in the neighborhood of the n-th order cluster is taken into account, the quantities $I_{1,2},\ \eta_{1,2}$ would no longer be integrals of motion. Still, the use of these variables as independent ones allows us to transfer from Hamiltonian, Eq.2, to a new one, $\mathcal{K}$, in terms of which the resonance $\vec{m}$ would be formally absent. According to Eqs.1, 3, on being put down in terms of new variables, the Eqs. of motion would retain their canonical nature:

$$\frac{dI_i}{d\theta} = -\frac{\partial\mathcal{K}}{\partial\eta_i}, \qquad \frac{d\eta_i}{d\theta} = \frac{\partial\mathcal{K}}{\partial I_i}, \qquad i = 1, 2 \quad (4)$$

$$\begin{aligned} \mathcal{K} &= \sum_{j=0}^{n} \mathcal{K}_{\vec{n},j} = \sum_{j=0}^{n} \sqrt{I_1^j I_2^{n-j}} \times \\ &\times \ \{\mathcal{F}_{\vec{n},j} \exp\{i[n\varepsilon_{n,j}\theta + j\eta_1 + (n-j)\eta_2]\} + \text{c.c.}\} \\ \mathcal{F}_{\vec{n},j} &= P_{\vec{n},k} \rho_{\vec{n},j} \exp\{i(n_x - n_y)(\arg P_{\vec{m},p})/2\} \\ \rho_{\vec{n},j} &= \sum_{l=l_{\min}}^{l_{\max}} (-1)^l \binom{n_y}{l} \binom{n_x}{j-l} \times \\ &\times \ (\cos\alpha)^{n_y+j-2l} (\sin\alpha)^{n_x-j+2l} \\ l_{\min} &= \max\{0; j - n_x\}, \quad l_{\max} = \min\{j; n_y\} \\ \varepsilon_{n,j} &= (\delta_x + \delta_y)/2 + \omega(1 - 2j/n) \end{aligned}$$

Any of $\mathcal{K}_{\vec{n},j}$ can result in an infinite increase of the total energy of oscillations, given certain resonant conditions are fulfilled. In absence of a difference resonance $\vec{m}$ there would have been a single straight line $\vec{n}\vec{\delta} = 0$ in the plane $\{\delta_x, \delta_y\}$. However, on this resonance being taken into account, a set of $(n+1)$ resonant curves $\varepsilon_{n,j} = 0$ emerge. These curves are given parametrically through Eqs. to follow,

$$\begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = -\frac{|P_{\vec{m},p}|}{\sin 2\alpha} \left\{ \begin{pmatrix} +1 \\ -1 \end{pmatrix} \cos 2\alpha + \begin{pmatrix} 1 \\ 1 \end{pmatrix} (1 - 2j/n) \right\}$$

Depending solely on the cluster order n, these curves form a family of hyperboles with asymptotes being given by $\delta_y + \delta_x + |\delta_y - \delta_x|(1 - 2j/n) = 0$. It can be easily found that the asymptotes coincide with resonant lines of all the n-th order sum resonances which may cross the cluster in question. Thus, the `switching-on' of the difference resonance yields splitting of the sum resonance's cluster.

Figs.1a,b, where Δ_ζ designates $\delta_\zeta/|P_{\vec{m},p}|$, show the splitting of sum resonances $\vec{n} = (1, 1)$ and $\vec{n} = (3, 0)$ driven by a skew field gradient and a normal sextupole, respectively. The width of split resonant lines is chosen as proportional to $\rho_{\vec{n},j}$.

III. COMPUTER SIMULATION

Assume the field imperfections that excite resonances $\vec{n}$ and $\vec{m}$ under study be localized at the quadrupole centers of all N cells of an accelerator. Suppose, that at these centers the following conditions are met: $\beta_{x,foc} = \beta_{y,def} = \beta_{\max}$, $\beta_{x,def} = \beta_{y,foc} = \beta_{\min}$, betatron phase advance between centers of adjacent F- and D-quads being the same. Take the following parameter values appropriate to the UNK case: $N = 160$, $r = 35$ mm, $\beta_{\max} = 152$ m, $\beta_{\min} = 32$ m. Distribute field imperfections along the lattice so as to excite in the vicinity of the working point only a sum resonance $\vec{n}$ and a linear difference resonance $\vec{m}$ with $\arg P_{\vec{n},k} = 0$ and $\arg P_{\vec{m},p} = 0$.

A. Joint Effect of $\vec{n} = (1, 1)$ and $\vec{m} = (-1, 1)$

Effect of these two resonances treated in variables $I_{1,2}, \eta_{1,2}$ results in emerging of three split resonant lines in plane $\{\delta_x, \delta_y\}$, see Fig.1a. Take the split resonance $j = 2$ and treat it separately from the others. In such a case

$$\mathcal{K} \simeq \mathcal{K}_{\vec{n},2} = -I_1\{|P_{\vec{n},k}| \cdot \sin 2\alpha\} \cos(2\varepsilon_{2,2}\theta + 2\eta_1) \quad (5)$$

and, thus, I_2 and η_2 are kept intact, i.e. $I_2(\theta) = I_2(0)$ and $\eta_2(\theta) = \eta_2(0)$. Put the working point exactly at the resonant curve $\varepsilon_{2,2}(\alpha) = 0$. Then Eqs.4, 5 with $\eta_1(0) = \pm\pi/4$ yield $\eta_1(\theta) = \eta_1(0)$ and total energy at $I_2(0) = 0$ varying in accordance with

$$I(\theta) = I(0)\exp\{\pm 2|P_{\vec{n},k}|\sin 2\alpha \cdot \theta\} \qquad (6)$$

For the preset values of $|P_{\vec{n},k}|$ and $|P_{\vec{m},p}|$ the skew gradient in l-th period is given by

$$\left(\frac{\Delta H_{x,1}\Delta s}{HRr}\right)_l = -\frac{2\pi/N}{\sqrt{\beta_{\max}\beta_{\min}}} \times \qquad (7)$$
$$\times\{|P_{\vec{m},p}| + 2|P_{\vec{n},k}|\cos(\frac{2\pi k}{N}(l-t))\}$$

where $t = 1/2$ and $t = 1$ for F- and D-quads, respectively. Fig.2 shows the values of $I(\theta)$ calculated via Eq.6, and via the numerical tracking during a hundred of turns with $|P_{\vec{n},k}| = 0.01, |P_{\vec{m},p}| = 0.1, I(0) = 1.0$ for $k = 73$, $p = 0$, $\alpha = 80°$, i.e. $\delta_x = 0.56713, \delta_y = 0.01763$. The agreement of results is fairly well.

B. Joint Effect of $\vec{n} = (3,0)$ and $\vec{m} = (-1,1)$

As in the previous example, for isolated split resonance $j = 3$ one has

$$\mathcal{K} \simeq \mathcal{K}_{\vec{n},3} = I_1^{3/2}\{|P_{\vec{n},k}| \cdot \cos^3\alpha\}\cos(3\varepsilon_{3,3}(\alpha)\cdot\theta + 3\eta_1)$$

Herefrom, $I_2(\theta) = I_2(0)$, $\eta_2(\theta) = \eta_2(0)$, and, given the position of the working point exactly at the resonance line $\varepsilon_{3,3}(\alpha) = 0$ with initial values $\eta_1(0) = \pm\pi/6$, one gets $\eta_1(\theta) = const$, while the expression for the total energy of oscillations at $I_2(0) = 0$ acquires the form of

$$I(\theta) = I(0)\cdot\{1 \mp 3\sqrt{I_0}|P_{\vec{n},k}|\cos^3\alpha\cdot\theta\}^{-2} \qquad (8)$$

For the preset values of $|P_{\vec{n},k}|$ and $|P_{\vec{m},p}|$ the distribution of the skew gradient is still given by Eq.7, provided a single difference resonance $\vec{m}$ is excited, while the distribution of quadratic nonlinearity acquires the form of

$$\left(\frac{\Delta H_{y,2}\Delta s}{HRr}\right)_l = \frac{48\pi}{N}\frac{|P_{\vec{n},k}|}{\beta^2_{\max} - \beta^2_{\min}} \times$$
$$\times\begin{cases} \sqrt{\beta^2_{\max}} & \cos\{\frac{2\pi k}{N}(l-\frac{1}{2})\} \quad \text{for F} \\ -\sqrt{\beta_{\max}\beta_{\min}} & \cos\{\frac{2\pi k}{N}(l-1)\} \quad \text{for D}\end{cases}$$

Fig.2b shows the values of the total energy $I(\theta)$ calculated with Eq.8 and numerical tracking results at $|P_{\vec{n},k}| = 0.01$, $|P_{\vec{m},p}| = 0.1$, $I(0) = 0.33$ for $k = 110$, $p = 0$ $\alpha = 80°$, i.e. $\delta_x = 0.56713, \delta_y = 0.01763$. The quantitative disagreement of theory against the numerical results can be accounted for by the second-order effect introduced by the quadratic nonlinearity which were omitted in the analytical treatment. Nevertheless, at least the qualitative agreement is still satisfactory.

IV. CONCLUSION

The presence of linear difference resonance changes the location and shape of resonant lines in plane of unperturbed betatron tunes. Namely, the cluster of sum resonances is split into family of hyperbolic resonant curves. These split resonances are capable of increasing the total energy of oscillations and, thus, can decrease the dynamic aperture, which has been found in Ref.[8].

References

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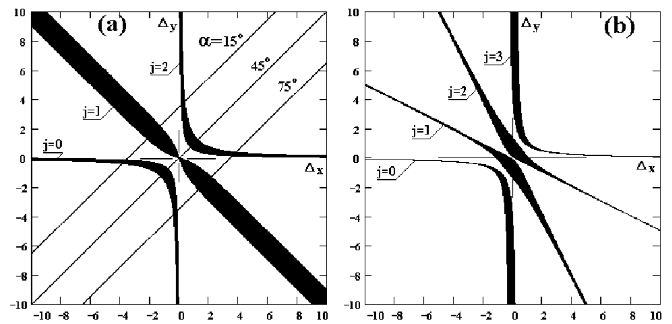

Figure 1. Splitting of sum resonances: $\vec{n} = (1,1)(a)$, $\vec{n} = (3,0)(b)$

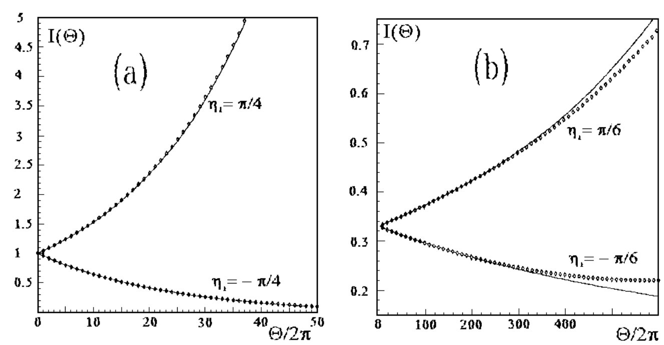

Figure 2. Comparison of theory (—) against tracking ($\diamond$) for $\vec{n} = (1,1)(a)$ and $\vec{n} = (3,0)(b)$