Self-consistent Analysis of Radiation and Relativistic Electron Beam Dynamics in a Helical Wiggler Using Lienard-Wiechert Fields

M. Tecimer and L. R. Elias, Center for Research and Education in Optics and Lasers (CREOL) and Physics Department
University of Central Florida, Orlando FL 32826

Lienard-Wiechert (LW) fields, which are exact solutions of the Wave Equation for a point charge in free space, are employed to formulate a self-consistent treatment of the electron beam dynamics and the evolution of the generated radiation in long undulators. In a relativistic electron beam the internal forces leading to the interaction of the electrons with each other can be computed by means of retarded LW fields. The resulting electron beam dynamics enables us to obtain three dimensional radiation fields starting from an initial incoherent spontaneous emission, without introducing a seed wave at start-up. In this paper, we present electromagnetic radiation studies, including multi-bucket electron phase dynamics and angular distribution of radiation in the time and frequency domain produced by a relativistic short electron beam bunch interacting with a circularly polarized magnetic undulator.

1. INTRODUCTION

The coherence characteristics of the radiation fields produced by a beam of relativistic electrons moving along a magnetic undulator depend on the degree to which electrons become organized under the influence of the ponderomotive force. Our approach shows that making use of the complete electric and magnetic fields produced by a point charge, the longitudinal beam dynamics of the particles is governed by the “near -” and “far zone” fields. Electric field components in the near zone are composed of terms falling off as R^{-2} and R^{-3} whereas in the “far zone” they vary as R^{-1} corresponding to radiation fields. The latter combined with the undulator fields gives rise to the ponderomotive force. For sufficiently high density electron beams, depending on the pulse length and axial charge distribution within a radiation wavelength, “near zone” fields have considerable effects on the longitudinal motion of the electrons and the associated bunching process, altering the characteristics of the produced radiation.

The purpose of this paper is to point out the way LW fields can be exploited in obtaining spectral and temporal behavior of the radiated fields for the self amplified spontaneous emission (SASE) process. In our formulation, rather than solving self-consistently the paraxial wave equation coupled to the relativistic single-particle equations of motion, we first compute retarded LW fields with few assumptions to drive the electron’s motion. Since the fields are evaluated in the time domain, the used approach allows their interaction with the electron beam with no restrictions on the frequency spectrum. Knowing all variables of the motion, such as retarded position, velocity, and acceleration of the charge, amplitude of the fields radiated by individual electrons in the beam are determined and summed at a observer surface far from the source. For simplicity, we consider in our simulations a filamentary sub-picosecond relativistic electron bunch which is substantially shorter than the slippage length.

Many simulation codes of free electron laser amplifiers utilize a single ponderomotive potential well imposing periodic boundary conditions at the bucket ends, thus neglecting slippage effects. Based on the formalism employed here, both the evolution of the multi-bucket electron phase space dynamics in the beam body as well as edges and the relative slippage of the radiation with respect to the electrons in the considered short bunch are naturally embedded into the simulation model.

A description of the particle and field dynamics underlying the code is outlined in section 2 followed in section 3 the numerical results demonstrating the evolution of the radiation in the time domain and its angular distribution. Here we study the evolution of the radiation for a monoenergetic beam with uniformly spaced electrons along the radiation wavelength as well as a pulse with shot noise in the electron phases.

2. PARTICLE AND FIELD DYNAMICS

The electric and magnetic fields produced by a point charge q moving along a trajectory $\vec{r}(t)$ with relativistic energy γmc^2 can be derived from the well known LW potentials. For the electric field strengths we have [1]:

$$\vec{E}(\vec{r}, t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{\hat{n}-\vec{\beta}}{\gamma^2(1-\vec{\beta}\cdot\hat{n})^3 R^2} + \frac{1}{c}\,\frac{\hat{n}\times\{(\hat{n}-\vec{\beta})\times\frac{d}{dt}\vec{\beta}\}}{(1-\vec{\beta}\cdot\hat{n})^3 R}\right]_{t_r} \quad (1)$$

All quantities in (1) have to be evaluated at the retarded time t_r The retardation condition $t_r = t-(R(t_r))/c$ connects the observation point time t to the source point time t_r where R represents the distance between the two points. The evolution of the three dimensional resultant radiation field can be determined when individual field contributions of all charges in the beam are superimposed in any point at the observer surface. Since LW fields are expressed in terms of particle’s retarded position and it’s time derivatives, relativistic Lorentz force equations for a coupled electron beam - radiation system have to be solved in a self-consistent way to describe completly particle’s motion. The transverse motion is almost entirely determined by the undulator magnetic field, whereas the axial motion of each electron is defined by the combined undulator and the resultant LW field produced by other electrons in the beam. The for the FEL mechanism crucial axial electron dynamics can be obtained directly from the equation for the energy exchange between the electron and LW fields. The equation for the energy change of the i_{th} electron :.

$$d\gamma_i/dt = (q/mc)\,\vec{E}\cdot\vec{\beta_i} \quad (2)$$

The electric field is a resultant field at the position of the ith electron obtained by summing up LW field contributions from the rest of the charges in the beam. The summation is implemented by including Doppler upshifted and downshifted parts of the radiation fields. The latter has, however, much smaller influence on the motion of the particles if the beam is highly relativistic.

To compute the field contribution of individual electrons to the summation, the retarded values of R, $\hat{n}$, $\vec{\beta}$, and $d\vec{\beta}/dt$ have to be determined from the present position of the particles in the undulator and be inserted into (1). The retarded distance R is given by

$$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

where x, y, z and x', y', z' represent the coordinates of the electron's present and retarded positions. At this point we introduce new dimensionless longitudinal position variables $\theta = k_u z$, $\theta' = k_u z'$, retarded distance $\rho = k_u R$, retarded axial distance $\chi_R = k_u(z_i - z'_j)$, and present axial distance $\chi_p = k_u(z_i - z_j)$ where prime denotes evaluation at retarded time and i, j refer to the particle's index in the beam. With the assumption of having a uniform longitudinal motion (β_z constant) during a numerical integration step for the energy exchange and employing the retardation condition, χ_R can be obtained by solving the equation:

$$\chi_R^2(1-\beta_z^2) - 2\chi_R\chi_p + \chi_p^2 - 2\left(\frac{a_u}{\gamma\beta_z}\right)^2(1-\cos\chi_R) = 0$$

The solution establishes the relation between the position of the source electrons at the time of emission and the position of the observer electron at the time of reception where the Lorentz force is exerted on that particular electron by the resultant field. Arranged in inverse powers of ρ, the rate of the energy exchange takes the form:

$$\begin{aligned}\frac{d\gamma}{d\tau} = r_0 k_u \sum_{j\neq i}\frac{1}{(1-\vec{\beta}_j\cdot\hat{n})^3}\Bigg[&\frac{1}{\rho^3}\{(a_u{}^2/\gamma^4\beta_{zj})(1+a_u{}^2)\sin\chi_R\\ &-(a_u{}^4/\gamma^4\beta_{zj})\sin\chi_R\cos\chi_R\ \}\\ &+\frac{1}{\rho^2}\{(\beta_{zi}/\gamma^2)(n_z-\beta_{zj})(1+a_u{}^2(1-\cos\chi_R))\\ &+(a_u/\gamma)^4(1+\sin^2\chi_R)-(a_u^2/\gamma^4)(1+a_u^2)\cos(\chi_R)\ \}\\ &-\frac{1}{\rho}\{(1-n_z\beta_{zj})(a_u/\gamma)^2\beta_{zj}\sin\chi_R\}\Bigg] \quad (3)\end{aligned}$$

where we introduced the dimensionless time variable $\tau = ck_u t$. ρ and the axial unit vector component n_z are given by:

$$\rho = \chi_R\sqrt{1+(a_u/\gamma\beta_z)^2\sin^2(\chi_R/2)/(\chi_R/2)^2},\quad n_z = \frac{\chi_R}{\rho}$$

respectively. The factor in the denominator, $1-\vec{\beta}\cdot\hat{n}$, can be expressed by $1-\vec{\beta}\cdot\hat{n} = 1 - n_z\beta_z - (a_u{}^2/\gamma^2\beta_z)\sin\chi_R/\rho$ Accounting for all forces involved in the electron-wave interaction, except for self radiation reaction, (3) gives a complete description of the change of electron's energy along the undulator. The term falling off as ρ^{-1} ($\sim R^{-1}$) contains the combined undulator-radiation fields causing modulation of the beam energy with subsequent bunching of the electron beam. The terms decreasing as ρ^{-2} and ρ^{-3} dominate at a distance close to the source. Thus we refer them as "near zone" fields. The factor $(1/\rho^2)(\beta_{zi}/\gamma^2)(n_z-\beta zj)/(1-\vec{\beta}_j\cdot\hat{n})^3$ describes the Coulomb interaction between the particles. It competes with the ponderomotive force reducing the depth of the ponderomotive potential well thus preventing particles from being trapped in the potential buckets. The rest of the "near field" terms are combined with the undulator fields and contributes to the energy modulation of the beam provided that the distance between the electrons are much smaller than the radiation wavelength. These fields would vanish in the abscence of an undulator. The numerical integration of (3) together with the relation $\beta_z = (1-(1+a_u^2)/\gamma^2)^{1/2}$ allows us finally to obtain the longitudinal motion of the electrons.

Having the knowledge of the position of each electron and it's time derivatives we can elaborate on the radiation evolution at a spherical surface of observation making use of the radiative part of (1). Superposition of fields emitted by each electron yields the resultant field:

$$\vec{E}_r(\vec{r},t) = \frac{q}{4\pi\varepsilon_0 c}\sum_i\left[\frac{\left(\hat{n}\cdot\frac{d}{dt}\vec{\beta}\right)_i(\hat{n}-\vec{\beta})_i - (1-\vec{\beta}\cdot\hat{n})_i\left(\frac{d}{dt}\vec{\beta}\right)_i}{(1-\vec{\beta}\cdot\hat{n})_i^3\ R_i}\right]_{tr}$$

Introducing the far-field assumption $R(t_r) \cong D - \hat{n}\cdot\vec{r}(t_r)$ where D is the distance from the observer to the entrance of the undulator and the unit vector $\hat{n}$ specifying the observation direction $\hat{n} = \{\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\}$, $1-\vec{\beta}\cdot\hat{n}$ and R(tr) may be approximated as:

$$R(t_r) = (D - z\cdot\cos\theta - (a_u/\gamma k_u\beta_z)\theta\sin k_u z) \quad \text{and}$$
$$1-\vec{\beta}\cdot\hat{n} = 1-(a_u/\gamma)\theta\cos k_u z - \beta_z\cos\theta$$

respectively. We use here the fact that the radiation pattern emitted by a charge in a helical undulator is azimuthally symmetric around z-axis, and confined to angles of the order of $\theta \approx (1/\gamma) \ll 1$. Noticing that $k_u l_b \ll 2\pi$ for the considered short bunch , where l_b is the bunch length, we can evaluate time structure and frequency composition of the radiated field components E_x and E_y at any point over the observer surface. The time evolution of the angular distribution is determined by $d^2W/d\Omega dt = \varepsilon_0 c|R\vec{E}|^2$ The total radiated power can be obtained performing the integral over the solid angle:

$$\frac{dW}{dt} \approx \varepsilon_0 c\int_0^{2\pi}\int_0^{\pi}|\vec{E}|^2R^2\theta d\theta d\phi \quad (4)$$

where $d\Omega \approx \theta d\theta d\phi$. The time integration of (4) over the radiation pulse yields the total radiated energy.

Evaluating Jackson's formula for the energy spectrum, we can also determine energy radiated per frequency ω, per solid angle Ω during the time the beam travels through the undulator:

$$\frac{d^2W}{d\Omega d\omega} = \frac{r_0 mc\omega^2}{4\pi^2}\left|\int_0^T\hat{n}\times\hat{n}\times\sum_i\vec{\beta}_i e^{i\omega\left(t-\frac{\hat{n}\cdot\vec{r}_i(t_r)}{c}\right)}dt\right|^2$$

where $\vec{r}_i$ and $\vec{\beta}_i$ are position and velocity of the i$^{\text{th}}$ electron obtained from the describtion of the self-consistent motion of electrons.

In order to give a correct account of the conservation of energy, energy loss of each electron due to its self field is included into (3). Using Lienard formula for the radiated power from a single electron we have:

$$P = \frac{q^2}{6\pi\varepsilon_0 c}\gamma^6\left(\left(\frac{d}{dt}\vec{\beta}\right)^2 - \left(\vec{\beta}\times\frac{d}{dt}\vec{\beta}\right)^2\right)$$

3. NUMERICAL RESULTS

The theoretical derivations presented in the previous section have been implemented in a simulation code to obtain angular and temporal characteristics of the three dimensional radiation fields emitted by a 35 MeV, 14 A, 70 μm subpico- second filamentary electron bunch propagating trough a 4.5 m long, 1.5 cm period helical undulator. The undulator parameter a_u is chosen to be 1.12. To the parameter set corresponding radiation wavelength is 3.5 μm . The code computes the time evolution of the three dimensional field amplitude with no initial bunching and seed wave at start-up. For a typical run ,a flat-top profile electron beam is used. The simulation particles are uniformly distributed along the electron pulse. The bunch , twenty radiation wavelength long , is much shorter than the slippage distance. To study effects of the shot noise in the electron phase , simulations with randomly spaced electrons are carried out.

4. REFERENCES

[1]J. D. Jackson, Classical Electrodynamics,p.657 (J. Wiley & Sons , 1975)

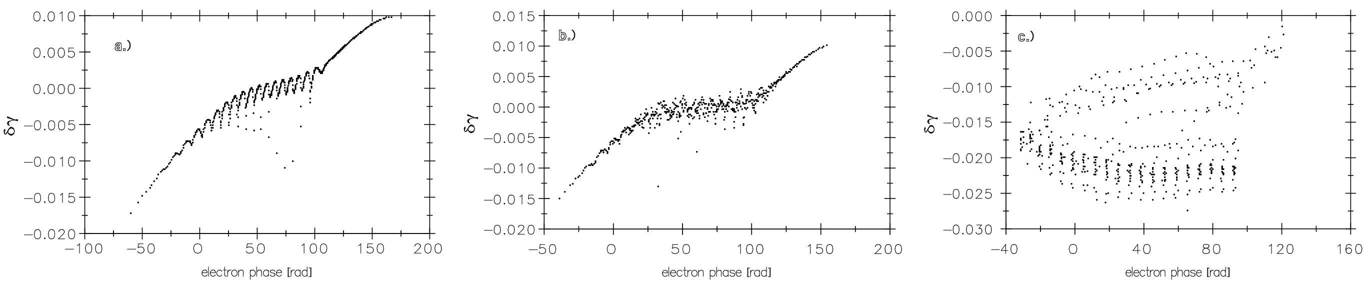
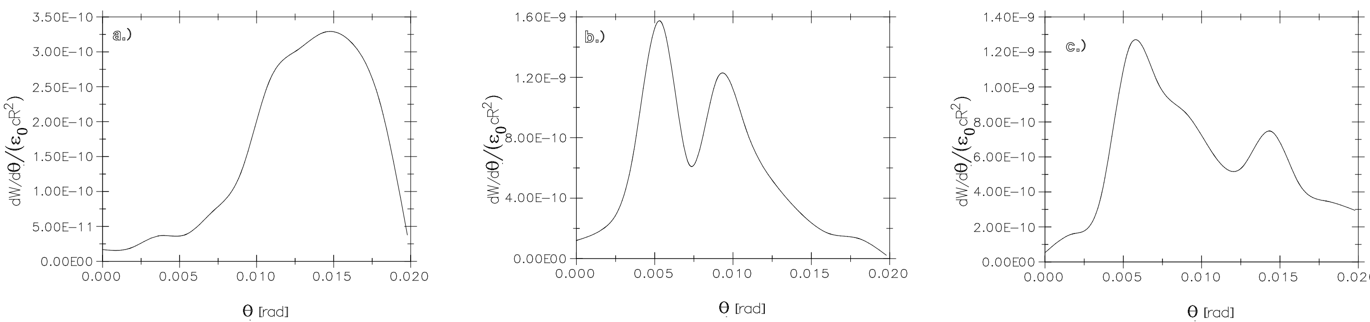
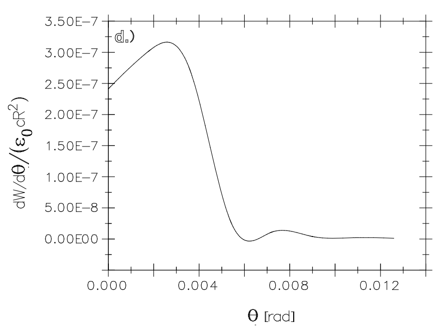


Fig. 1 a-c. show phase versus exchanged energy of initially uniformly and randomly distributed monoenergetic electron beam after traveling trough a 4.5 m long helical undulator. Plot 1.c is obtained from the numerical integration of (19) by considering the 1/R dependent (ponderomotive force) term alone.In plot 1.a-b the influence of the “near-zone” fields on the longitudinal beam dynamics is included. In Plot 1.b the periodicity of the potential wells is perturbed due to the initial random distribution of the charge along the bunch length.





Figures 2. a-c illustrate the angular distribution of the radiated energy by the initially randomly phased electron bunch. The angular distribution observed at a surface far from the source is divided into three different time domains; leading - and trailing edge of the radiation pulse, figures 2a. ,2 c respectively, and the steady state regime fig. 2b, where θ is the observation angle.Due to the (partial) bunching of the electron beam at the end of the undulator, the coherent superposition of radiation pulses from all electrons in the beam produce, as a result of wave interference effects a narrower cone of radiation than the radiation generated by the electrons at the entrance of the undulator.Dependent on the distribution of the electron phases, by summing the amplitude of the fields radiated by individual electrons one can obtain interferece patterns at the observer surface where constructive (destructive) interference takes place for certain observation angles.Fig.2d. shows the angular distribution of the radiated energy in the absence of the “near-zone” fields . Due to the increased coherence in the resultant field, the angular radiation cone becomes narrower than the ones shown in figures 2.a-c.