On Landau Damping of Collective Beam-Beam Modes

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Abstract
In this paper we report a simplified model, where collective beam-beam oscillations can be described by a differential equation similar to the Schrödinger equation in quantum mechanics. In this case, the stability criteria can be obtained inspecting the behaviour of effective potential well curves.

I. INTRODUCTION

It is well known, that a study of collective beam-beam instabilities typically demands a solution of a very complicated system of integral equations. Usually, if we neglect a nonlinearity of the beam-beam kicks, produced by the stationary motion of the colliding bunches, these equations predict (see, for instance, [1]) a resonance instability of the betatron coherent oscillations with increments of the order of $\omega_0\xi$, where ω_0 is the revolution frequency, and ξ beam-beam strength parameter. Since such an initial assumption eliminates Landau damping of unstable modes, we may expect that such models overestimate the strength of the collective beam-beam instability.

Here we briefly report a model, which enables the evaluation of the effect of the Landau damping on the collective beam-beam modes. More detailed calculations can be found in [2]. For the sake of simplicity, we assume one interaction point (IP), identical colliding bunches and zero dispersion function at the IP.

II. SHORT BUNCHES

The description of Landau damping of the collective beam-beam modes can be simplified within the framework of the model described in [1]. Namely, we calculate the eigenmodes of the horizontal coherent oscillations, assuming that colliding bunches have very flat unperturbed distribution functions in amplitudes of betatron oscillations

$$f_0(J_x, J_z) = N\delta(J_z)F_0(J_x)/\epsilon. \tag{1}$$

If F_0 is a Gaussian distribution, ϵ has the sense of the bunch horizontal emittance, N is the number of particles in the bunch. We consider, first, the case when the lengths of colliding bunches $\sigma_s \ll \beta_x^*$, where β_x^* is the value of the horizontal β-function at the IP. Then, the incoherent horizontal betatron oscillations of a particle in the interaction region (IR) are given by

$$\begin{aligned} x = \sqrt{J_x}\cos\psi_x, \quad R_0(p_x/p) = dx/d\tau = x', \\ \tau = \omega_0 t, \quad \psi_x' = \nu_x + \Delta\nu_x(J_x). \end{aligned} \tag{2}$$

Here, $2\pi R_0$ is the perimeter of the orbit and p is the momentum of the particle, $\Delta\nu_x(J_x)$ is the incoherent beam-beam tune shift. Near an isolated resonance ($\nu_x + \Delta\nu_x(J_x) = n/m$) a linearized system of Vlasov's equations written for the horizontal coherent betatron oscillations ($f^{(1,2)} = f_0 + \delta f^{(1,2)}$, $f_0 \gg |\delta f^{(1,2)}|$)

$$\delta f^{(1,2)}(J_x, \psi_x, \tau) = \sum_{m_x \neq 0} f_{m_x}^{(1,2)}(J_x) e^{im_x\psi_x - i\nu\tau} \tag{3}$$

yields the following system of integral equations [1]

$$f_{m_x}^{(1,2)} = \frac{m_x\xi}{\nu - m_x\Delta}\int_0^\infty du' u' F_0'(u') G(u,u') X_m^{(2,1)}(u'). \tag{4}$$

Here, $u^2 = J_x/\epsilon$, $\xi = Ne^2/(2\pi pc\epsilon)$, $X_m^{(1,2)} = f_{m_x}^{(1,2)} + f_{-m_x}^{(1,2)}$, $F_0' = dF_0/d(u^2/2)$, $\Delta = \nu_x + \Delta\nu_x(J_x) - n/m$ and

$$G(u,u') = \int_{-\infty}^{\infty} \frac{dk}{|k|} J_{m_x}(ku) J_{m_x}(ku'), \tag{5}$$

where $J_m(x)$ is the Bessel function. The calculation of the integral in Eq.(5) results in [3] ($m = |m_x|$)

$$G(u,u') = \frac{1}{m}\begin{cases} (u/u')^m, & u \le u', \\ (u'/u)^m, & u > u'. \end{cases} \tag{6}$$

As can be seen, Eq.(4) separates the so-called π- and o-modes (the sign $(-)$ corresponds to π-mode)

$$\chi^{\pm} = X_m^{(1)} \pm X_m^{(2)}, \tag{7}$$

which satisfy independent equations

$$\chi^{\pm} = \frac{\pm 1}{m}\int_0^\infty du' u' V(u') G(u,u') \chi^{\pm}, \tag{8}$$

where

$$V(u) = \frac{2\xi\Delta(u)}{(\nu/m)^2 - \Delta^2(u)}\frac{dF_0}{d(u^2/2)}, \quad \mathrm{Im}\nu > 0. \tag{9}$$

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The substitution $\chi^{\pm} = w_{\pm}/\sqrt{u}$ and subsequent double differentiation of Eq.(8) over u transform it into the following differential equation:

$$w''_{\pm} + \left[\pm 2V(u) - \frac{m^2 - 1/4}{u^2}\right] w_{\pm} = 0. \qquad (10)$$

Eq.(10) can be used to study stability of various stationary distributions F_0. For instance, for a water-bag distribution function when $dF_0/du = -2\delta(u-1)$ and $(\Delta_0 = \nu_x - n/m)$

$$\Delta(u) = \Delta_0 + \xi' \begin{cases} 1, & u \leq 1, \\ 1/u, & u > 1 \end{cases}, \quad \xi' = \frac{Ne^2}{\pi p c \epsilon}, \qquad (11)$$

Eq.(10) results in

$$w_{\pm}(u) = \begin{cases} u^{1/2+m}, & u \leq 1, \\ u^{1/2-m}, & u > 1, \end{cases} \quad 1 = \frac{\mp 2\xi'\Delta/m}{(\nu/m)^2 - \Delta^2}. \qquad (12)$$

These equations predict unstable coherent oscillations within the stopbands $|\Delta \mp \xi'/m| \leq \xi'/m$ with the maximum increments ($\mathrm{Im}\nu = \xi'$) independent of the mode number m_x. Note, that in these equations Δ is calculated at $u = 1$. This fact causes general shifts of the centers of the stopbands when ξ' increases.

For a more realistic Gaussian distribution when

$$\frac{dF_0}{d(u^2/2)} = -e^{-u^2/2}, \quad \Delta = \Delta_0 + \frac{2\xi[1 - \exp(-u^2/2)]}{u^2},$$

Eq.(10) cannot be solved directly. However, in the case of unstable oscillations ($\mathrm{Re}\nu = 0$), general properties of eigenfunctions and spectra can be predicted using the analogy of Eq.(10) and the Schrödinger equation, which is written for a particle with zero energy moving in effective potential well

$$U_{eff}(u) = \frac{m^2 - 1/4}{u^2} \mp 2V(u). \qquad (13)$$

Since $U_{eff}(u)$ is a real function and, therefore, the operator in Eq.(10) is a self-adjoined one, Eq.(10) can have non-trivial solutions, if a potential curve ($U_{eff}(u)$) has a negative minimum ($dU_{eff}(u_0)/du = 0$, $U_{eff}(u_0) < 0$) between the stop-points ($U_{eff}(u_{1,2}) = 0$, $u_1 < u_0 < u_2$). As seen from Eqs(9) and (13) for π-modes U_{eff} can be negative, if $\Delta_0 < 0$, and for o-modes, if $\Delta_0 > 0$. This determines a usual location of the stopbans of π- and o-modes relative to the point $\Delta_0 = 0$. Stability of coherent oscillations can be studied inspecting the behaviour of U_{eff}. For example, Fig.1 shows the possibility for unstable solutions with increments of 0.7ξ for dipole oscillations at least within the stopband $-2 \leq \Delta_0 \leq -1$. Fig.2 shows an increase in the depth of U_{eff} for dipole oscillations if $\mathrm{Im}\nu$ decreases. This figure also shows that slow modes ($\mathrm{Im}\nu \ll \xi$) can penetrate in the core of the bunch which, generally, may cause stronger perturbations of incoherent oscillations. Fig.3 illustrates the possibility for unstable sextupole modes with increments of 0.08ξ within the stopband $-0.8 \leq \Delta_0 \leq -0.7$. Figs1–3 indicate that Landau

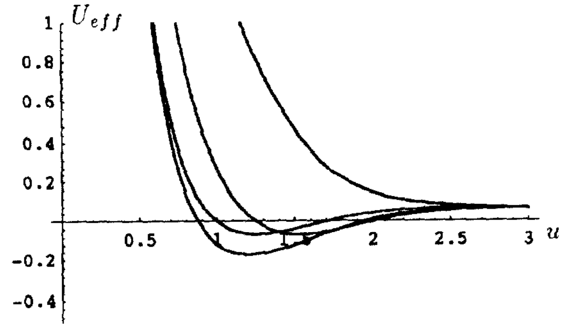

Figure 1: U_{eff} vs u; $m = 1$, $\nu = 0.7i\xi$, from top to bottom: $\Delta_0/\xi = -0.5, -1, -2, -1.5$.

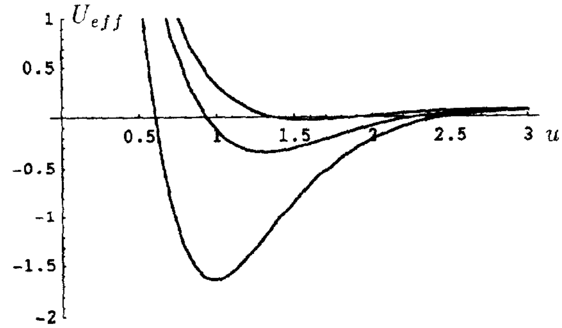

Figure 2: U_{eff} vs u; $m = 1$, $\Delta_0 = -\xi$, from bottom to top: $\nu/i\xi = 0.25, 0.5, 0.75$.

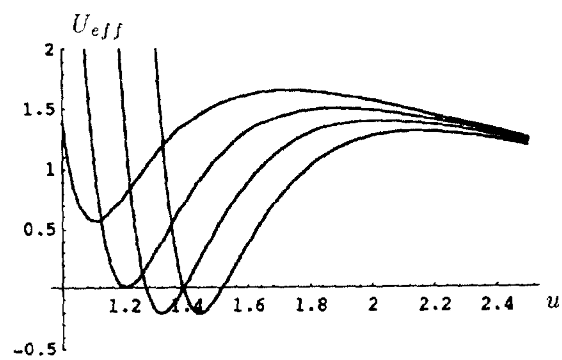

Figure 3: U_{eff} vs u; $m = 3$, $\nu = 0.08i\xi$, from top to bottom (at $u = 1.6$): $\Delta_0/\xi = -0.85, -0.8, -0.75, -0.7$.

damping due to a non-linearity of the beam-beam kick does not stabilize at least the dipole, quadrupole and sextupole beam-beam modes of short bunches [2]. This result was recently confirmed [4] by the numerical solution of Eqs(4).

III. LONG BUNCHES

As was shown in [5] the collective beam-beam instability of long bunches (in our model, $\sigma_s \simeq \beta_x^*$) can be strongly suppressed by the so-called phase-averaging effect [6]. This suppression occurs due to strong modulation of the β-function and of the phase of betatron oscillations in IR:

$$\begin{gathered} x = \sqrt{J_x\beta(\tau)}\cos\psi_x(\tau), \\ \beta(\tau) = \beta_x^* + \frac{(R_0\tau)^2}{\beta_x^*}, \quad \psi_x(\tau) = \arctan(s/\beta_x^*). \end{gathered} \tag{14}$$

For coherent oscillations the effect of the bunch length is described by a simple redefinition of the beam-beam parameter [5]

$$\xi \to \xi_{eff} = \xi Y_m(\zeta), \quad \zeta = \frac{\sigma_s}{\beta_x^*}. \tag{15}$$

For this reason, for long bunches Eq.(9) takes the form:

$$V(u) = \frac{2\xi Y_m\Delta(u)}{(\nu/m)^2 - \Delta^2(u)}\frac{dF_0}{d(u^2/2)}, \quad \mathrm{Im}\nu > 0. \tag{16}$$

The suppressing factors (Y_m) in Eqs(15) and (16) depend on the ratio of β^* to β_x^* and on the mode-number (m). As an illustration, we can assume, for instance, $\beta^* = \beta_x^*$ when [5]

$$Y_m = \int\limits_{-\infty}^{\infty} \frac{du}{\sqrt{\pi}} e^{-u^2} \left(\frac{1+i\zeta u}{1-i\zeta u}\right)^m . \tag{17}$$

Figs4 and 5 show that, in the region $\sigma_s \simeq \beta_x^*$, the phase-

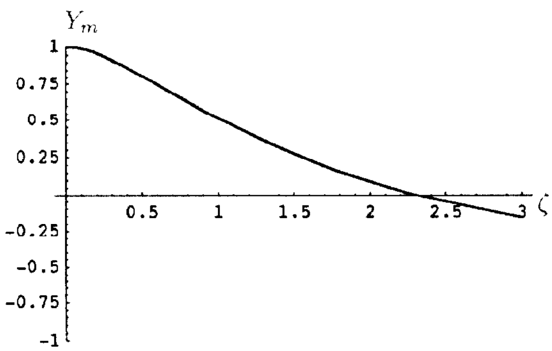

Figure 4: Dependence of Y_m on the bunch length; $m = 1$.

averaging effect roughly twice decreases the value of $V(u)$ for dipole modes and, practically, eliminates the instability of the sextupole modes. Note, that the dependence of Y_m on the bunch length for the sextupole (as well as for higher modes [5]) is rather sharp. This means that a suppression of the higher-order coherent resonances can occur

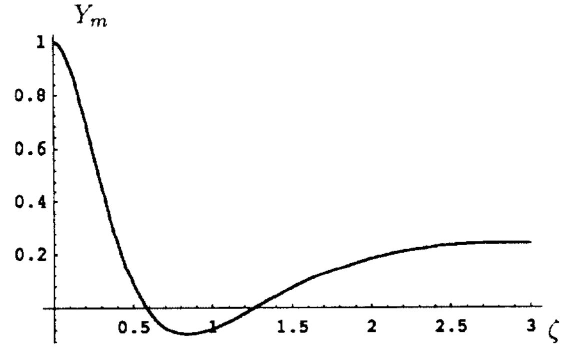

Figure 5: Dependence of Y_m on the bunch length; $m = 3$.

for rather short bunches. As seen in Fig.5, the strength of the sextupole resonance decreases twice when $\sigma_s \simeq 0.25\beta_x^*$.

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IV. REFERENCES

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