On Solvable Model with Synchrotron Mode-Coupling

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Abstract

Typically the study of the collective stability of a bunch with the strong coupling of the synchrotron modes demands the solution of the infinite set of integral equations. The paper reports two cases, when these mode-coupling equation can be solved directly for synchrobetatron and for synchrotron collective modes.

I. INTRODUCTION

It is well known that in many cases the possibility of increasing the beam current in a storage ring is limited by coherent interaction of the beam bunches with their environment. In the case of a single-turn interaction, when the bunch wakes decay faster than the revolution period in the ring, the specific features of coherent instabilities significantly depend on the ratio of the bunch coherent frequency shift Ω_m to the frequency of synchrotron oscillations of particles in this bunch ω_s. If this ratio is high, the calculation of the increments of coherent modes and stability criteria demands the solution of a system of integral equations, which generally couple the harmonics of the bunch distribution function over the phases of synchrotron oscillations (see, for instance in [1,2]). The solvable examples of such problems except for their heuristic worth can be used to test the codes, designed for numerical study mode-coupling problems.

Here we report a simple model, when these equations can be solved directly for the synchrobetatron and synchrotron modes. However, the resulting dispersion equations are very complicated and, except for the case of a weak mode-coupling, still require a numerical solution.

II. SYNCHROBETATRON OSCILLATIONS

We describe the unperturbed vertical betatron and synchrotron oscillations of a particle near the closed orbit by usual formulae:

$$\begin{gathered} z = a_z\cos\psi_z, \quad \theta = \omega_s t + \varphi, \varphi_s = \varphi\cos\psi_s, \\ \dot\varphi = -\omega_s\varphi_s\sin\psi_s, \quad \dot\psi_x = \omega_x = \omega_0\nu_x, \\ \dot\psi_s = \omega_0\nu_s, \quad I_z = \frac{p}{2R_0}\nu_z a_z^2, \quad I_s = \frac{\mathcal{E}\nu_s}{2\omega_0\alpha}\varphi_s^2. \end{gathered} \tag{1}$$

Here, $\Pi = 2\pi R_0$ is the perimeter of the orbit, ($\mathcal{E} \simeq pc$) is the energy of a particle. We neglect Landau damping due to the nonlinearity of the particle synchrotron oscillations and we do the calculations for the case of the dipole betatron coherent oscillations, which are described by the expansion of the distribution function ($f_0 = F_0(I_z)\rho(\varphi)$)

$$f = f_0 + \sqrt{I_z}e^{i\psi_z}\sum_{m_s=-\infty}^{\infty}\chi_m(\varphi)e^{im_s\psi_s - i\omega t} + c.c. \tag{2}$$

Then, for a single-turn interaction the amplitudes χ_m satisfy the system of integral equations ($\omega = \pm\omega_z + \Delta\omega_m$) [2]

$$\begin{gathered} (\Delta\omega_m - m_s\omega_s)\chi_m = \rho(\varphi)\int_{-\infty}^{\infty} dn\Omega_{m,n}J_{m_s}(n\varphi)\chi(n), \\ \chi(n) = \sum_{m_s=-\infty}^{\infty}\int_0^{\infty} d\varphi\varphi J_{m_s}(n\varphi)\chi_m(\varphi). \end{gathered} \tag{3}$$

Here, the bunch wake is described by the value $\Omega_{m,n}$, giving the coherent frequency shift of the coasting beam. Eqs(3) can be solved exactly for the simplified model, where ($\Omega_m = m_z\Omega$, $m_z = \pm 1$)

$$\rho(\varphi) = \delta(\varphi_0^2 - \varphi^2), \quad \Omega_{m,n} = \frac{i\Omega_m}{\pi(n + i\Delta)}. \tag{4}$$

In the region $|\Delta\omega_m| \ll \omega_s$ the quantity Ω defines the coherent frequency shift of the betatron mode ($m_s = 0$). According to Eqs(3) and (4) we write $\chi_m(\varphi) = C_m\delta(\varphi_0^2 - \varphi^2)$, which replaces Eqs(3) by an equivalent system of the algebraic equations

$$\begin{gathered} (\Delta\omega_m - m_s\omega_s)C_m = \frac{i\Omega_m}{\pi}\sum_{m'_s=-\infty}^{\infty} Q_{m,m'}C_{m'}, \\ Q_{m,m'} = \begin{cases} -i\pi, & m_s = m'_s = 0, \\ \dfrac{4}{\pi}\dfrac{\sin(\pi[m - m'])}{m^2 - m'^2}, & m_s, m'_s \neq 0\ . \end{cases} \end{gathered} \tag{5}$$

Using $x = \Delta\omega_m/\omega_s$, $w = \Omega_m/\omega_s$, we rewrite Eqs(5) in the following form ($W = 4w/\pi^2$)

$$(x - w)C_0 = iW\sum_{k=0}^{\infty}\frac{C^-_{2k+1}}{(2k+1)^2}, \quad C^+_{2p} = \frac{2iWxS^-}{x^2 - 4p^2}, \tag{6}$$

$$C^-_{2k+1} = \frac{2iWx}{x^2 - (2k+1)^2}\left[\frac{C_0}{(2k+1)^2} + S^+\right]. \tag{7}$$

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Here,

$$S^{+} = \sum_{p=1}^{\infty} \frac{C_{2p}^{+}}{(2k+1)^2 - 4p^2},$$
$$S^{-} = \sum_{k=0}^{\infty} \frac{C_{2k+1}^{-}}{(2k+1)^2 - 4p^2}, \quad (8)$$
$$C_{2k}^{+} = C_{2k} + C_{-2k}, \quad C_{2k+1}^{-} = C_{2k+1} - C_{-(2k+1)}.$$

Substituting C_{2p}^{+} from Eqs(6) into Eq.(7), we obtain

$$C_{2k+1}^{-} = \frac{2iWx}{x^2 - (2k+1)^2} \left[\frac{C_0}{(2k+1)^2} + \sum_{k'=0}^{\infty} S_{k,k'} C_{2k'+1}^{-} \right],$$
$$S_{k,k'} = \sum_{p=1}^{\infty} \frac{2iWx}{[x^2 - 4p^2][(2k+1)^2 - 4p^2][(2k'+1)^2 - 4p^2]}. \quad (9)$$

The calculation of the sum over p in $S_{k,k'}$ results in:

$$S_{k,k'} = -\frac{1}{2x^2(2k+1)^2(2k'+1)^2} + \frac{\pi}{4x} \frac{\cot(\pi x/2)}{[x^2 - (2k+1)^2][x^2 - (2k'+1)^2]}.$$

Using this expression, we find

$$C_{2k+1}^{-} = -\frac{2W^2}{(2k+1)^2[x^2 - (2k+1)^2]} \frac{w}{x - w} \times \sum_{k'=0}^{\infty} \frac{C_{2k'+1}^{-}}{(2k'+1)^2} - W^2 \frac{\pi x \cot(\pi x/2)}{[x^2 - (2k+1)^2]^2} S^{-}. \quad (10)$$

The solutions of Eqs(10) read:

$$C_{2k+1}^{-} = \frac{A(x)/(2k+1)^2}{x^2 - (2k+1)^2} + \frac{B(x)}{[x^2 - (2k+1)^2]^2}. \quad (11)$$

The substitution of C_{2k+1}^{-} from Eq.(11) in Eq.(10) yields the dispersion equation

$$1 - \frac{x}{w} = 2W^2 F_1(x) - \frac{2W^4 F_2^2(x) \pi x \cot(\pi x/2)}{1 + W^2 \pi x \cot(\pi x/2) F_3(x)}, \quad (12)$$

where

$$F_1(x) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4[x^2 - (2k+1)^2]},$$
$$F_2(x) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2[x^2 - (2k+1)^2]^2}, \quad (13)$$
$$F_3(x) = \sum_{k=0}^{\infty} \frac{1}{[x^2 - (2k+1)^2]^3}.$$

III. SYNCHROTRON OSCILLATIONS

Similar model can be used to describe the mode-coupling instability of the coherent synchrotron oscillations, if we take as $\rho(\varphi)$ the so-called water-bag distribution

$$\rho(\varphi) \propto \begin{cases} 1, & \varphi \le \varphi_0 \ , \\ 0, & \varphi > \varphi_0 \end{cases}$$

and the longitudinal wake from a pure resistive impedance, which does not depend on n. With these assumptions the synchrotron collective modes are defined by the system of equations, which can be written in the form, similar to that of Eqs(3)

$$(x - m_s)\chi_m = i m_s w \delta(\varphi^2 - \varphi_0^2) \int_{-\infty}^{\infty} \frac{dn J_{m_s}(n\varphi)\chi(n)}{n + m_s \nu_s + i\Delta}, \quad (14)$$

or, after the substitution $\chi_m = C_m \delta(\varphi - \varphi_0)$,

$$(x - m_s) C_m = i m_s w \sum_{m'_s} C_{m'} \int_{-\infty}^{\infty} \frac{dn J_{m_s}(n) J_{m'_s}(n)}{n + m_s \nu_s \varphi_0 + i\Delta}. \quad (15)$$

Since typically $\nu_s \varphi_0 \ll 1$, we can expand the integrand in Eq.(15) in the power series of $m_s \nu_s \varphi_0$. Taking into account in Eq.(15) the first two terms of this expansion, we obtain

$$(x - m_s) C_m = i m_s w \sum_{m'_s} Q_{m,m'} C_{m'} - i w m_s^2 \nu_s \varphi_0 \sum_{m'} C_{m'} \int_{-\infty}^{\infty} \frac{dn J_{m_s}(n) J_{m'_s}(n)}{n^2}. \quad (16)$$

The second term in this equation couples the modes with the same parity. In the region $|w| \ll 1$, when the mode-coupling is negligible small, this term gives a leading contribution in the decrements (or increments) of the synchrotron modes

$$-\,\mathrm{Im} x = \delta/\omega_s \simeq w \nu_s \varphi_0. \quad (17)$$

The oscillations will be unstable, when $w < 0$.

On the contrary, in the region $|w| \sim 1$, the leading contribution in the r.h.s. of Eq.(16) gives the first term, while the second describes small perturbations. Neglecting in this equation the values, proportional to $\nu_s \varphi_0 \ll 1$ and using the definitions from Eq.(8), we rewrite Eq.(16) in the following form

$$C_{2p}^{+} = \frac{8Wp^2 S^{+}}{x^2 - 4p^2}, \quad C_{2k+1}^{-} = \frac{2Wx(2k+1)S^{-}}{x^2 - (2k+1)^2}. \quad (18)$$

Substituting here the first equation into the second, we find

$$C_{2k+1}^{-} = \frac{4xW^2(2k+1)}{x^2 - (2k+1)^2} \sum_{k'=0}^{\infty} \Sigma_{k,k'} C_{2k'+1}^{-}, \quad (19)$$

where

$$\Sigma_{k,k'} = \sum_{p=1}^{\infty} \frac{1}{[(2k+1)^2 - 4p^2][(2k+1)^2 - 4p^2]} - x^2 S_{k,k'}.$$

Now, simple calculations result in the dispersion equation

$$1 = -W^2 \pi x^2 \cot \frac{\pi x}{2} \sum_{k=0}^{\infty} \frac{2k+1}{[x^2 - (2k+1)^2]^3}. \quad (20)$$

V.V. Danilov and E.A. Perevedentsev are acknowledged for their stimulating discussions. I indebted to K. Hirata, S. Kurokawa, N. Yamamoto and K. Yokoya for their valuable help during the preparation of this report. It is my pleasure to thank KEK and its B-factory group for their hospitality.

IV. REFERENCES

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