Passage through a Half-Integer Resonance due to Space Charge for Different Initial Distributions

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I. Introduction

The problem of space charge continues to be interesting in connection with the development of Kaon and Neutron Facilities and Superconducting Super Collider Projects. In the first, a high average proton beam intensity can be reached by super-high peak currents in each acceleration cycle, so a painting procedure is required. In colliders, as is well known, super luminosity can be reached by minimizing the emittance growth during acceleration. In other words, the task consists of accelerating beam with the maximum attainable current density and with minimum achievable emittance growth. This is why this task arises when new accelerator projects are developed.

At least two possibilities exist how to investigate the space charge effects at the design stage: analytical and numerical simulation [1–3]. Each of them has its own advantages and disadvantages. Numerical simulation using the macro-particle approximation is very well developed but requires a lot of computer time and so little flexibility in variation of the initial conditions. Because of this numerical methods are restricted in their optimization capability.

Analytical investigations gives a clearer picture of the physical phenomena but the necessarily simplified the physical model sometimes gives an incorrect interpretation of the results. Analytical research can be divided in two types: the envelope equation method for self-consistent distributions and the nonlinear equation for higher distribution moments . In our work we use numerical simulation as the base to achieve the correct results, but for interpreting the results, the simplified analytical model is used as well.

II. Problem Statement

In almost any accelerator emittance growth is observed during beam injection. This is due to resonance crossing and mismatch between the beam and the acceptance. But this growth depends on what distribution is chosen during injection. In this paper we study the emittance growth under resonance crossing for different initial beam distributions. To eliminate the reasons for emittance growth we consider a linear lattice without errors. In this case the emittance should grow only because of the intrinsic resonances, or in other words, the envelope oscillation.

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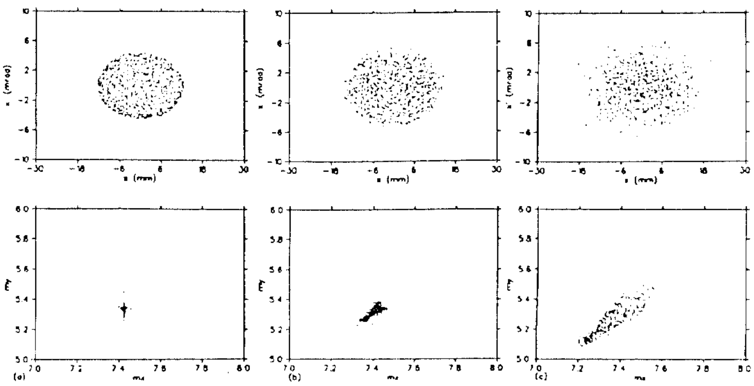

Figure 1: Phase space and tune diagram for K-V (a), waterbag(b) and Gaussian (c) distributions.

III. Numerical Calculation

The problem is solved for three different initial distributions: K-V, waterbag and Gaussian. To simulate self-consistent beam motion a two-dimensional tracking program is used. Space-charge forces are found by solving the Poisson equation with zero boundary conditions. Longitudinal motion effects and beam bunching are not taken into consideration. The particle oscillation frequency is found from the rotation angle in the normalized phase space. As a test lattice, we use the racetrack structure for TRIUMF Booster with two arcs and working point 7.65/5.6 although these results could be useful for the SSC LEB and AGS as well.

The initial phase space projections and the advanced phase diagram are shown in Figs. 1(a), 1(b) and 1(c) for K-V, waterbag and Gaussian distributions respectively. For each distribution we take the same *rms* emittance.One can see that all distributions have the same centroid, indicating the same coherent tune shift.

Figure 2 shows the emittance growth versus the number of turns for different distributions. The maximum emittance growth (a factor 2.8) was observed for K-V and waterbag at only a factor 1.25 for the Gaussian distribution.

Figures 3(a), 3(b) and 3(c) show the phase space for all observed distributions after five turns. It is obvious that

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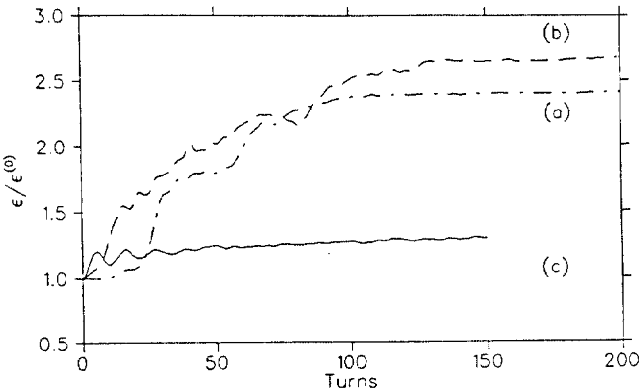

Figure 2: Emittance growth vs the turn number for K-V (a), waterbag (b) and Gaussian (c) distributions.

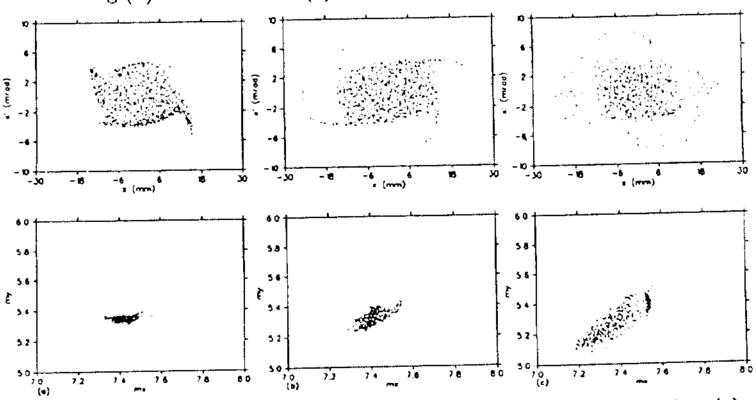

Figure 3: Phase space for K-V (a), waterbag (b) and Gaussian (c) distributions after 5 turns.

the beam passes through half-integer and fourth-order resonances. In all cases the amplitude of oscillation is finite.

Figures 4(a), 4(b) and 4(c) show the distributions after 45 turns. For waterbag and K-V distributions we can observe a half-integer resonance which is stabilized by a fourth-order resonance. For the Gaussian distribution we observe the distribution stabilized by a fourth-order resonance.

Figures 5(a), 5(b) and 5(c) show the final distributions for the same cases. The instability does not grow. They are stabilized, but the particles are still in the half-integer resonance excited by the envelope oscillation. This self-stabilization phenomenon is known for nonlinear system,

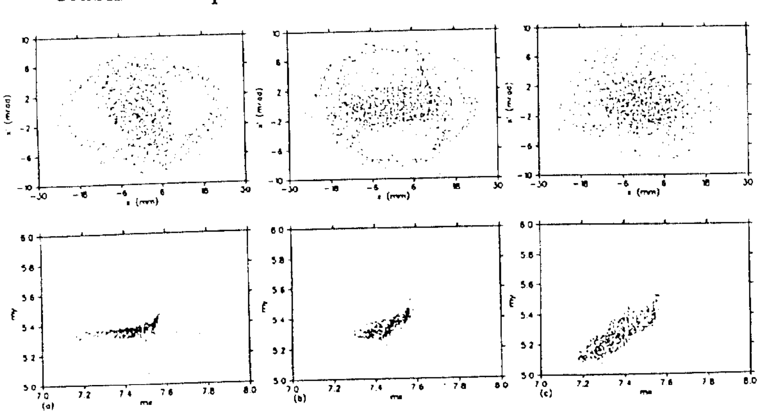

Figure 4: Phase space for K-V (a), waterbag (b) and Gaussian (c) distributions after 45 turns.

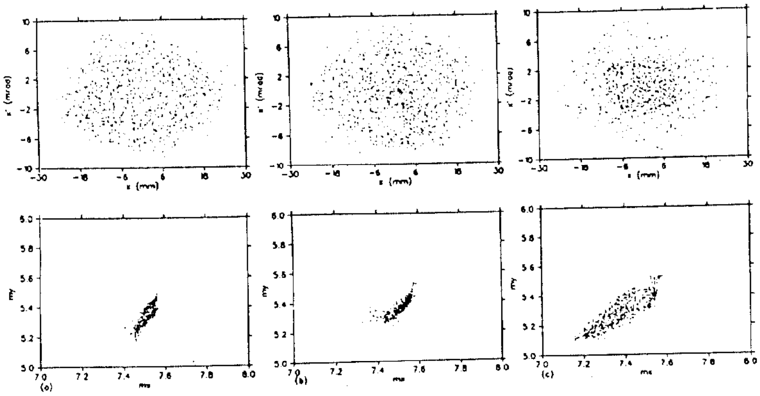

Figure 5: Stabilized distribution for initial K-V. (a), waterbag (b) and Gaussian distributions.

but in the space-charge problem we observe a system where the nonlinearity changes with time.

We now try to explain the phenomenon using a simplified analytical model which describes the resonance interaction.

IV. Analytical Description

Taking into account that we treated the two planes separately in the numerical experiment, the analytical study has been performed for a symmetrical cylindrical beam. The Maxwell equation for the symmetrical beam with density distribution ρ is:

$$\frac{1}{r}\frac{\partial(rE_r)}{\partial r} = \frac{1}{\varepsilon_0}\rho(r) \tag{1}$$

We choose the m-order binomial distribution, which includes in itself the three types of distribution considered: Kapchinskij-Vladimirskij (m=0), the waterbag (m=1) and the Gaussian ($m \to \infty$):

$$\rho(r) = \frac{m}{\pi b^2}[1 - \frac{r^2}{b^2}]^{m-1}, \tag{2}$$

where b is the maximum size of the beam. Then the dispersion function σ is defined as $\sigma = \frac{b}{\sqrt{2m+2}}$ and the rms emittance $\epsilon_{rms} = \sigma^2/\beta$. The equation of motion for any particle will be:

$$\frac{d^2r}{ds^2} + \mathcal{K}(s)^2 r - \frac{r_0 N}{\pi R \beta_0^2 \gamma^3}\frac{r}{2\sigma^2}\mathcal{F}(r) = 0, \tag{3}$$

or using the smooth approximation $r = \eta\sqrt{\beta}$ and the new longitudinal coordinate $d\theta = \nu_0 \beta ds$

$$\eta'' + \nu_0^2\eta = 2\nu_0\frac{r_0 N}{4\pi\beta_0^2\gamma^3\epsilon_{rms}}\eta[1 + \sum_p b_p \cos p\theta]\mathcal{F}(\eta), \tag{4}$$

where we use the Fourier expansion for $\beta = \overline{\beta}(1 + \sum_p b_p \cos p\theta)$ and the mean β-function $\overline{\beta} = \frac{R}{\nu_0}$, R being the average radius of the accelerator.

$$\mathcal{F}(r) = \begin{cases} 1/2 & \text{for K-V} \\ \frac{2}{3} - \frac{2}{9}\frac{1}{2!}\frac{\eta^2}{2\epsilon_{rms}} & \text{for waterbag} \\ 1 - \frac{1}{2!}\frac{\eta^2}{2\epsilon_{rms}} + \frac{1}{3!}(\frac{\eta^2}{2\epsilon_{rms}})^2 - \dots & \text{for Gaussian} \end{cases} \tag{5}$$

Usually one denotes the value $r_0 N/4\pi\beta^2\gamma^3\epsilon_{rms}$ as the tune shift $\delta\nu$. Bogolubov's method can be used to solve this nonlinear equation:

$$\eta = \sqrt{\varepsilon}\cos\Phi$$
$$\eta' = -\nu_0\sqrt{\varepsilon}\sin\Phi, \tag{6}$$

or passing to new variables

$$\varepsilon = \eta^2 + (\frac{\eta'}{\nu_0})^2$$
$$\Phi = -\arctan(\eta'/\nu_0\eta). \tag{7}$$

Differentiating the new variables ε and Φ with respect to θ and substituting (6), we get:

$$\frac{d\varepsilon}{d\theta} = -2\delta\nu_0\varepsilon\sin 2\Phi[1+\sum_p b_p\cos p\theta]\mathcal{F}(\varepsilon)$$
$$\frac{d\Phi}{d\theta} = \nu_0 - 2\delta\nu\cos^2\Phi[1+\sum_p b_p\cos p\theta]\mathcal{F}(\varepsilon) \tag{8}$$

The function $\mathcal{F}(\varepsilon)$ has the same meaning as (6), but with a new argument $\frac{\varepsilon}{2\epsilon_{rms}}\cos^2\Phi$. In the absence of any resonances we can average (9) over the whole cross section of the beam:

$$\overline{\frac{d\varepsilon}{d\theta}} = 0$$
$$\frac{\overline{\Phi}}{d\theta} = \nu_0 - \Delta\nu, \tag{9}$$

where $\Delta\nu$ is the average tune shift. It may be sham that equals $\frac{5}{8}\delta\nu$ for Gaussian, $\frac{5}{9}\delta\nu$ for waterbag and $\frac{1}{2}\delta\nu$ for K-V. This means that in a resonanceless system ($b_p = 0$) the emittance does not grow and the coherent tune shift does not depend on the distribution with the same *rms* emittance. On other hand the tune spread is a maximum for the Gaussian and equals zero for the K-V.

In fact, for any lattice the Fourier expansion of the envelope β-function involves in itself all harmonics which could give the resonance condition for emittance growth. Consider the case when $\mathcal{F}$ involve just two terms and can be represented as $\mathcal{F} = f_0 - \varepsilon f_1\cos^2\Phi$. Then

$$\overline{\frac{d\varepsilon}{d\theta}} = \delta\nu\varepsilon b_m f_0\sin(2\Phi - m\theta) -$$
$$\delta\nu\varepsilon^2 f_1 b_m[\frac{1}{2}\sin(2\Phi - m\theta) - \frac{1}{4}\sin(4\Phi - m\theta)]$$
$$\frac{\overline{\Phi}}{d\theta} = \nu_0 - \delta\nu f_0 + \frac{3}{4}\delta\nu\varepsilon f_1 + \frac{1}{4}\delta\nu f_0 b_m\cos(2\Phi - m\theta) -$$
$$\delta\nu\varepsilon f_1 b_m[\frac{1}{8}\cos(4\Phi - m\theta) + \frac{1}{2}\cos(2\Phi - m\theta)], \tag{10}$$

where m is the resonant harmonic number with amplitude b_m. One can see from these equations that any square nonlinearity in distribution gives the half-integer and the fourth-order resonances simultaneously. They can exist only together. The main question is only whether the fourth-order resonance stabilizes the half-integer resonance. The trajectories of particles on the phase plane are described by the equation:

$$\frac{1}{2}\delta\nu b_m f_0\varepsilon\cos 2\Psi - \frac{1}{4}\delta\nu b_m f_1\varepsilon^2\cos 2\Psi$$
$$-(\nu_0 - \delta\nu f_0)\varepsilon - \frac{3}{8}\delta\nu f_1\varepsilon^2 - \frac{1}{4}\delta\nu f_0 b_m\varepsilon\cos 2\Psi +$$
$$\frac{1}{4}\delta\nu f_1 b_m\varepsilon^2\cos 2\Psi = C(\varepsilon, \Psi), \tag{11}$$

where $\Psi = \Phi - m\theta/n$ is the "slow phase" in the n-th order resonance. The terms with ε^2 stabilize parametric resonance. So for the distribution where the square term is absent, the half-integer resonance will give sufficient growth. Of course we should explain here that our analytical model doesn't allow us to take into account the self-consistent redistribution, which could create the term with ε^2 for any distribution after passing through the resonance. At least this explanation gives the answer why the emittance grows so strongly in the resonance for the K-V distribution.

Using more high-order terms of the binomial distribution it is possible to show that any order resonance will be stabilized by the next order distribution. Since the envelope β-function has all harmonics we will always have the resonant condition for one or more harmonics. So maximum stabilization will be observed for the Gaussian, where any perturbation will be distributed over an unlimited number of harmonics.

V. Conclusion

We have studied in this paper a beam passing through the grid of intrinsic envelope resonances, where the half-integer resonance is maximum. At the initial stage a beam with any kind of distribution is very sensitive to the half-integer and fourth-order resonances. During 60-100 turns the distribution becomes self-stabilized and very similar to Gaussian. Maximum growth is observed for maximum uniform distribution in real space. For intrinsic resonances selfstabilization occurs when most particles remain in resonance.

In conclusion we would like to thank Dr. A.Iliev for helpful discussions during this work.

VI. References

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