# Simulation of Space-Charge Dominated Beam Dynamics in an Isochronous AVF Cyclotron

Shane R. Koscielniak, TRIUMF Vancouver B.C. Canada
Stefan R. Adam, PSI Villigen Switzerland.

*Abstract*

This paper describes a beam dynamics code 'PICN' for simulation of high current proton beams in an isochronous azimuthally-varying-field cyclotron. It is assumed that the median plane and vertical motions are decoupled, so that the internal space-charge forces can be calculated by a particle-in-cell method applied to the median plane charge distribution. We assume operation in a regime where vertical focusing is not weak. This paper details the simulation method and approximations, including a summary of the equations of motion. Results of the code, when applied to the P.S.I. Injector II cyclotron are presented, and compared with results from an earlier code that treated the beam as composed of a distribution of rigid spheres.

## I. Introduction

As has been noted by several authors [1] [2] [3], space-charge is important in isochronous cyclotrons for the following reason: there is no longitudinal focusing and there is strong radial-azimuthal coupling. Because the space-charge electric fields are severely non-linear, practical investigation must proceed by numerical methods [4] [5].

The original code 'PICS' [3] considered the beam to be formed of charged spheres, and completely neglected the internal motions within these spheres. The main argument to justify this simplification was that the betatron oscillations are much faster movements than the deformations of a beam bunch due to space charge forces.

The new simulation code, to be called PICN, assumes that the radial and the vertical betatron motions are decoupled, just as in the code PICS. However, the median plane internal motions within a charged sphere are now to be included explicitly. The sphere is decomposed into cylinders and the cylinders are divided into vertical rods or needles. There are now new freedoms in the motion: rods within the same initial sphere need not have the same centre nor the same oscillation frequency, and individual time-dependent modulations of tune and betatron amplitude are now allowed. The new model assumes that all needles have the same, fixed height; i.e. the spheres are replaced by full cylinders. The artificial force law used in PICS is replaced by the force between two uniformly charged vertical rods.

### A. Elementary force law between two needles

Consider two parallel rods of length $2b$ separated by a distance $a$ with charges $Q_1$ and $Q_2$ respectively. Then the mutual repulsive force is

$$F(a,b) = \frac{Q_1 \times Q_2}{2\pi\epsilon_0\, a\,(2b)^2}\left[\sqrt{(2b)^2+a^2} - a\right] .$$

## II. Motion equations in the laboratory frame

The centre of charge (and mass) moves with velocity $\mathbf{u}$. An arbitrary particle in the bunch has some velocity $\mathbf{v}$. We should like to find an equation for $\frac{d}{dt}(\mathbf{v}-\mathbf{u})$. We use the energy equations to eliminate the time derivatives of the $\gamma$-factors from the momentum equations. For the reference particle: $m_0\gamma_u d\mathbf{u}/dt = \mathbf{F}_0^{ex} - \mathbf{u}(\mathbf{u}\cdot\mathbf{F}_0^{ex})/c^2$. For the general particle: $m_0\gamma_v d\mathbf{v}/dt = \mathbf{F}_2^{sc} + \mathbf{F}^{ex} - \mathbf{v}[\mathbf{v}\cdot\mathbf{F}^{ex}]/c^2$. where $\mathbf{F}_2^{sc} = \mathbf{F}^{sc} - \mathbf{v}[\mathbf{v}\cdot\mathbf{F}^{sc}]/c^2$. $\mathbf{F}^{ex}$ is an externally applied force due to magnets and cavities, say. $\mathbf{F}^{sc}$ is the force due to space-charge; i.e. from the whole assembly of which the 'test' particle is a member.

### A. An expression for space-charge force

The space-charge term can be expressed in terms of the Coulombic electric fields, due to the assembly of particles, as measured (or calculated) in the frame co-moving with the centre of mass of the group. We now define $\|$ and $\perp$ to mean parallel and perpendicular to $\mathbf{u}$. Let the electric field be $\mathbf{E}'_{sc}$ in a frame which is co-moving with $\mathbf{u}$. It is assumed that in this rest frame (of the bunch) there is an electric field $\mathbf{E}'_{sc}$ but no magnetic field $\mathbf{B}'_{sc}$. The electric field is resolved into components transverse and longitudinal to the reference motion: $\mathbf{E}' = \mathbf{E}'_\perp + \mathbf{E}'_\|$. In the laboratory frame the space-charge force on a test particle is:

$$\mathbf{F}^{sc} = q\left[\mathbf{E}'_\| + \gamma_u\mathbf{E}'_\perp + \mathbf{v}\wedge(\mathbf{u}\wedge\mathbf{E}'_\perp)\gamma_u/c^2\right] . \qquad (1)$$

The vector $\mathbf{v}\wedge(\mathbf{u}\wedge\mathbf{E}'_\perp)$ occurring in the space-charge force term, $\mathbf{F}^{sc}$ equation (1), is perpendicular to $\mathbf{v}$ and so the magnetic field due to the beam cannot alter the energy.

We now substitute $\mathbf{v} = \mathbf{v}_\| + \mathbf{v}_\perp = \mathbf{u} + \Delta\mathbf{v}$ and note that

$$\frac{1}{q}\mathbf{F}_2^{sc} = (\mathbf{E}'_\| + \gamma_u\mathbf{E}'_\perp)\left(1 - \frac{uv_\|}{c^2}\right) - \frac{\Delta\mathbf{v}}{c^2}(v_\| E'_\| + v_\perp\gamma_u E'_\perp) \qquad (2)$$

is an exact expression. Now, to first order $(1 - uv_\|/c^2) \approx 1/\gamma_u \approx 1/\gamma_v$. Also note that the final term of (2) in $\Delta\mathbf{v}/c^2$ is negligible. Hence, now approximately,

$$\mathbf{F}_2^{sc} = (q/\gamma_u)\left[\mathbf{E}'_\perp + \mathbf{E}'_\|/\gamma_u\right] .$$

### B. Explicit Representation

We acknowledge the cylindrical symmetry of the cyclotron applied $\mathbf{E}^{ex}$ and $\mathbf{B}^{ex}$ fields, and adopt cylindrical polar coordinates $(\rho, \phi, z)$ for the reference particle position $\mathbf{x}_0 = \rho\,\mathbf{e}_\rho(\phi)$. We note that with space-charge there are preferential directions parallel and perpendicular to $\mathbf{u}$,

and take a local cartesian (rectangular) coordinate system $(r, s, z)$ for the position vector of a general particle, $\mathbf{x} = \mathbf{x}_0 + \mathbf{e}_\rho r + \mathbf{e}_\phi s + \mathbf{e}_z z$. The velocity increment is $\Delta\mathbf{v} = (\dot{r} - s\dot{\phi})\mathbf{e}_\rho + (\dot{s} + r\dot{\phi})\mathbf{e}_\phi + \mathbf{e}_z \dot{z}$ .
$\phi$ is a reference coordinate, and we specify this to be isochronous so that it rotates at constant angular velocity $\dot{\phi} = \omega_c = (q/m_0)B_c^0$. In this case,

$$\Delta\dot{\mathbf{v}} = \mathbf{e}_\rho[\ddot{r} - 2\dot{s}\omega_c - \omega_c^2 r] + \mathbf{e}_\phi[\ddot{s} + 2\dot{r}\omega_c - \omega_c^2 s] + \mathbf{e}_z \ddot{z} .$$

We must compare the above identity for acceleration with the equation for forces. For brevity, we set $(q/m_0)$ equal to unity. For simplicity we write the form appropriate to no external electric field $\mathbf{E}^{ex} = \mathbf{0}$.

$$\gamma_u \Delta\dot{\mathbf{v}} = [\Delta\mathbf{v} \wedge \mathbf{B}_0 + \mathbf{v} \wedge \Delta\mathbf{B}]^{ex} + (1/\gamma_u)\left[\mathbf{E}'_{\|}/\gamma_u + \mathbf{E}'_{\perp}\right]^{sc} .$$

Here $\mathbf{B}_0^{ex} = (B_\rho = 0,\ B_z^0,\ B_\phi = 0)$ is the reference field in the medium plane, and $\Delta\mathbf{B}$ the field increment at $(r, s, z)$.

### C. Almost Flat field, $\Delta\boldsymbol{B} = \boldsymbol{0}$

For the almost flat magnetic field $B_z^0(\rho) = \gamma(\rho)B_c^0$, where $B_c^0$ is the magnetic field at the cyclotron centre, there is a single Lorentz force term expressible as $\Delta\mathbf{v}\wedge\mathbf{B}_0^{ex}$.

Since time and turns accumulate equally, there is a simple transformation between derivatives. Let $\theta = \omega_c t$, so one turn corresponds to $\theta = 2\pi$. Under the approximation $\gamma_u = \gamma(\rho)$, the motion equations become:

$$\begin{aligned} (r' - s)' &= (1/\omega_c^2)(q/m_0)\, E_r'(r,s,z)/\gamma_u^2 \\ (s' + r)' &= (1/\omega_c^2)(q/m_0)\, E_s'(r,s,z)/\gamma_u^3 , \end{aligned}$$

where the superfix prime denotes the derivative with respect to radian-turns.

### D. An approximation for AVF

It is no simple matter to find equations of motion in the median plane so as to describe a smooth focusing due to the combination of radial field gradients and sectored azimuthal variations of the magnetic field.

Let the angle formed between radius vectors to the general and reference particles be $\theta$. We take a force proportional and perpendicular to the 'extra' azimuthal velocity $\Delta v_\theta = \dot{s}\cos\theta - \dot{r}\sin\theta$ and to the 'extra' radial velocity $\Delta v_\rho = \dot{r}\cos\theta + \dot{s}\sin\theta$, at point $(s, r)$ . Thus we take a focusing force $\mathbf{F}_{focus} = m[\mathbf{e}_\rho \Delta v_\theta - \mathbf{e}_\theta \Delta v_\rho]$. In terms of the cartesian unit vectors $\mathbf{e}_r, \mathbf{e}_s$, we find $\mathbf{F}_{focus} = m(\mathbf{e}_r s' - \mathbf{e}_s r')$, and the space-charge term $\mathbf{F}_2^{sc} = \mathbf{e}_s F_s + \mathbf{e}_r F_r$. Hence the equations of motion are:

$$(s' + \nu r)' = F_s \ , \qquad (r' - \nu s)' = F_r \ . \qquad (3)$$

The radial betatron tune $\nu$ is given by $\nu = (m+1)$. There are two constants of motion if $F_s = F_r = 0$; however the speed is not exactly conserved. These equations yield betatron motion in the form of circles in the median plane, which property lends itself to finding self-consistent matched charge distributions under internal space-charge forces. Further, these equations are formally identical with those presented by Kleeven [6].

## III. Starting ensemble

The overall density of points (in real space) will be the convolution of an elementary disc composed of concentric rings of short vertical rods convolved with the distribution of disc centres, which may be distributed homogeneously over a rectangular grid. Each particle point $[s, r, s', r']$ may carry a different charge.

### A. Matched elementary disc – no space charge

The elementary ensembles are circular in position and in velocity space, and made as follows. From a uniformly populated disc (in $r, s$-space) generate the correlated velocities for betatron motion according to $r' = \nu s$ and $s' = -\nu r$.

### B. Matched elementary disc – with space-charge

It is desirable that elementary charge clouds be stationary under the action of the internal space-charge force; this facilitates comparison with the sphere model in PICS.

The disc consists of concentric rings, and our matching scheme will be to adjust the velocity coordinates of particles on each ring so as to give a self-consistent distribution under space-charge. We take a system of local polar coordinates $(\rho, \theta)$ with centre at the reference particle, such that $s = \rho\cos\theta$ and $r = \rho\sin\theta$. The equations (3) become:

$$\rho'' - \rho(\theta')^2 + \nu(\rho\theta') = F_\rho \ ; \ -2\rho'\theta' - \rho\theta'' + \nu\rho' = -F_\theta . \ (4)$$

Here $F_\rho$, $F_\theta$ are the radial and azimuthal (with respect to centre of cloud) components of $\mathbf{F}_2^{sc}$ . We look for an equilibrium circular solution of (4) with $\theta'$, $\rho = constants$. For the rods at radius, $\rho$, the angular velocity $\theta'$ is given by:

$$2\theta'(\rho) = \nu + \sqrt{\nu^2 - 4F_\rho/\rho} .$$

The matched disc is found as follows. (i) Generate a uniformly populated disc. (ii) Numerically solve for the space-charge forces $F_s$, $F_r$ and transform to $F_\rho$, $F_\theta$. (iii) Ring by ring evaluate $\theta'(\rho)$. (iv) Generate the correlated velocities according to $r' = \theta' s$ and $s' = -\theta' r$.

## IV. Example cases

The mutual forces between the elementary charge discs are not compensated for and will cause a perturbation of the matched circular betatronic motion and also cause the cloud centres to move. These effects were studied for a coasting beam in PSI Injector II. All cases are for a 1 mA, 5 MeV beam with vertical height 2 mm. The integration algorithm is a fourth order time explicit Runge-Kutta integrator, used with a 0.05 turns integration step size and the program working fully in double precision.

Fig. 1: A round starting distribution, not matched under space-charge, of one centre and 5000 needles in a single cloud remains round, but "breathes", expanding and shrinking; it performs a monopole mode oscillation. Plotted are the r.m.s. radial and azimuthal width as a function of turn number. The oscillation frequency is lower than $\nu_\rho$ due to space charge forces. This beam has a longitudinal extension only from coupled radial-longitudinal betatron oscillations, but *no* phasewidth.

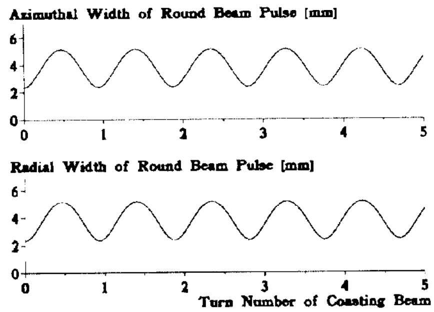

Fig. 1: Initial unmatched, round ensemble.

Fig. 2: Radial and azimuthal r.m.s. width of a bunch of 7.5 degree initial phase width, showing the increase of the radial width and the decrease of the azimuthal width followed by a small, slow rise towards a common value for both widths at a round beam (seen from the top); rather a 'disk' than a ball as the height is 2 mm.

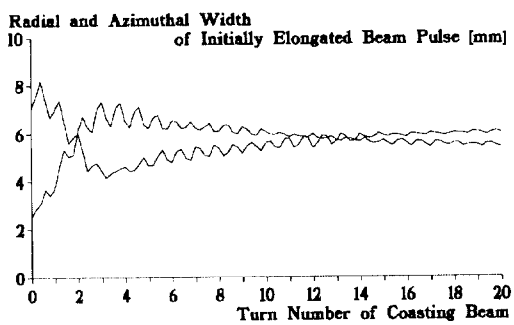

Fig. 2: Initial unmatched, oblong ensemble.

The initial increase of radial width and decrease of azimuthal width arises from rotation of the bunch. At an angle of 90° of the rotation, the beam is wider in the radial direction than in the azimuthal one. As this rotation carries on, it slows down with the approach of the two widths towards the matched case at about 75% of the initial length. The initial rotation is faster than the 15° case.

Fig. 3: The PICN simulation shows five successive top views of a bunch with initial phase width 15° (Injector II is on 10th harmonic) on turn numbers 0, 2, 4, 6, 8.

Fig. 4: The same case as PICN, but with the earlier program PICS. It really looks quite similar to previous case; the basic mechanism of forming an S-shape and then gradually a galaxy-like distribution is present in both models. However, the deformation is generally about 20 to 25 percent weaker for the needle model PICN.

Since it is clear that the phenomena are basically mismatching, the interesting question is whether a round charge distribution (not necessarily a sphere) is stable (i.e. matched). The test was a 'flying saucer' with an r.m.s width of 8 mm azimuthally and radially. This case, Fig. 5, shows almost constant radial and azimuthal width over many turns. From the way the centres are initialized it was a little 'squarish' at the beginning, and so probably this case did not have a precise matching; therefore it shows a small partial monopole mode as well.

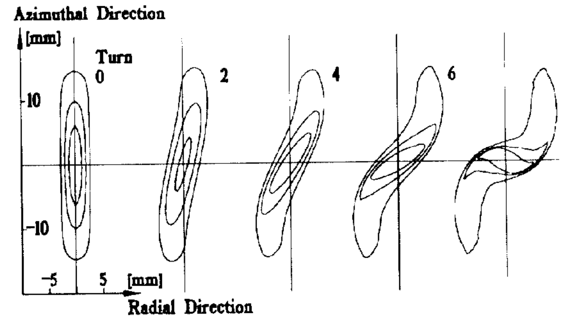

Fig. 3: PICN simulation of 15° phase width beam.

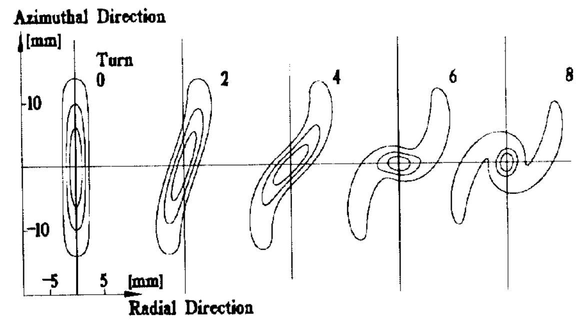

Fig. 4: PICS simulation of 15° phase width beam.

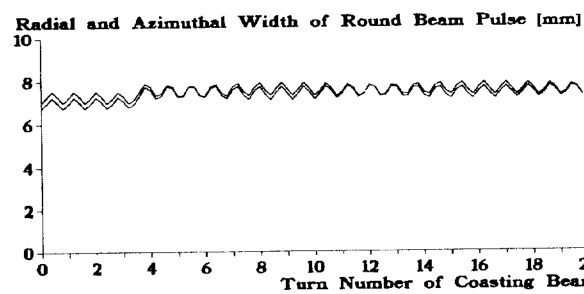

Fig. 5: Initial matched, round ensemble.

## V. References

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