A Matrix Theory of the Motion of an Ellipsoidal Bunch in a Beam Control System with a Rectlinear Optical Axis and with Space Charge

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Abstract

The motion of a 3-dimensional ellipsoidal bunch of charged particles in an arbitrary external electromagnetic field is considered taking into account the effect of space charge. The first approximation of the electromagnetic field within the bunch due to space charge has been found. The nonlinear matrix equation in the envelope variable space is written. An effective recursive computational method for the solution of the nonlinear beam envelope evolution is proposed.

1 Introduction

A time-varying field can be used for the transverse focusing of particles and in some cases a high-frequency focusing system can be constructed more simply and cheaply than a strong-focusing system [1]. The possibility to use a high-frequency focusing field in nuclear microprobes [2] opens up possibilities for the analysis of high-frequency or rapidly moving processes. Most advanced accelerator and microprobe applications require high-brightness beams [3]. In some cases -like a narrow beam through the whole system or a very bright beam- the beam is dominated by space-charge forces throughout the entire system from source to target.

In this paper the electromagnetic field inside a beam bunch due to the space-charge is presented. The paraxial, nonlinear equations of motion of this bunch in an arbitrary external electromagnetic field, as well as a new, recursive technique for solving these equations, have been constructed and are presented. A more detailed report on this theory is to be found in [4]. In two recent papers [5] and [6], this theory has been applied to a special case with an infinitely long beam (i.e. a DC-beam) and with an elliptical beam cross-section in a static electromagnetic field.

2 The coordinate system and notations

The motion of the particles of a bunch is described relative to a single particle, *the reference particle*, which follows a *reference trajectory*. The equations of motion and of the 4-vector electromagnetic potential A are written in a coordinate frame x, moving with the reference particle. All symbols and notations in this paper will follow the same conventions as used in ref. [7]. The coordinates of the reference particle are described by the four vector z_m. This is chosen such that $z_{m1} = z_{m2} = 0$, $z_{m4} = ct_m$. In this paper we restrict ourselves to first-order (paraxial) focusing with a rectilinear reference trajectory $z = z(t_m)$. An arbitrary particle is described by a 4-vector x, where the components of x are the deviation of any particle from the reference particle [7]. Here x_1 and x_2 are the transverse coordinates, x_3 is the longitudinal coordinate and $x_4 = c(t-t_m)$, i.e. the time coordinate. All the particles in a bunch are detected at the same time $t = t_m$. This means that the observer is located in the plane $x_4 = 0$ and that $z_4 = z_{m4} = ct_m$.

The quantities B, E, ρ, I and j denote magnetic field induction, electric field strength, charge density, beam current and current density, respectively. The quantities p and γ denote momentum and total energy of the reference particle. The 3-dimensional vectors B and E are expressed in the inverse of the units used to measure x and z $\left(i.e.\ m^{-1}\right)$. The 4-dimensional vector A and the quantities γ, p and I are dimensionless. The quantities ρ and j are expressed in m^{-2}.

As notated above, $x_4 = 0$. Therefore, instead of the notation $\overline{x}$ for the three dimensional vector, we will use for convinience the notation x, $i.e. \overline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x$.

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The following notations are also used:

$$x' = \frac{dx}{dz_{m4}},\ d\{x\} = dx_1 dx_2 dx_3,\ d\{x'\} = dx'_1 dx'_2 dx'_3$$

$$z'_1 = x'_1,\ z'_2 = x'_2,\ z'_3 = z'_{3m} + x'_3,\ z'_4 = 1$$

3 THE ELECTROMAGNETIC POTENTIAL A INSIDE THE ELLIPSOIDAL BUNCH DUE TO THE SPACE CHARGE

Let us consider the motion of a bunch of particles which is the canonical central, 3-dimensional ellipsoid $\Omega(x)$. This ellipsoid is described by the following equation:

$$0 \le \alpha(x) = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} \le 1 \qquad (1)$$

Here a_1, a_2 and a_3 are functions of z_{m4}.

The 4-vector potential A is written in the form:

$$A(x, z_{m4}) = \int_{\Omega(x^*)} \frac{j(\overline{z}(x^*), z_{m4} - \tilde{x}^* x^*)\, d\{x^*\}}{4\pi\sqrt{(\tilde{x} - \tilde{x}^*)(x - x^*)}}$$

We suppose that the bunch of particles in the space (x') is also the canonical, central, 3-dimensional ellipsoid. Therefore $A_1 = 0$ and $A_2 = 0$.

Let a, b, c be any three unequal, positive, real numbers which we suppose to be arranged in descending order of magnitude:

$$a > b > c > 0$$

where:

$$a = \max_i a_i,\ c = \min_i a_i,$$

$$b = a_i,\ if\ a_i \ne a\ and\ a_i \ne c$$

Assuming that $\rho = \rho_m$ is constant inside the ellipsoid we obtain the components A_3 and A_4 in the following forms [8]:

$$A_3 = \frac{p}{\gamma}\rho_m \psi,\ A_4 = \rho_m \psi$$

where:

$$\psi = \frac{1}{2}\left(M_0 - M_1 x_1^2 - M_2 x_2^2 - M_3 x_3^2\right)$$

$$M_0 = \frac{abc}{\sqrt{a^2 - c^2}} F(\phi, k)$$

$$M_1 = \frac{abc}{(a^2 - b^2)\sqrt{a^2 - c^2}}\left[F(\phi, k) - E(\phi, k)\right]$$

$$M_2 = -\frac{abc}{\sqrt{a^2 - c^2}} F(\phi, k) + \frac{abc\sqrt{a^2 - c^2} E(\phi, k)}{(a^2 - b^2)(b^2 - c^2)} - \frac{c^2}{b^2 - c^2} \qquad (2)$$

$$M_3 = -\frac{abc}{(b^2 - c^2)\sqrt{a^2 - c^2}} E(\phi, k) + \frac{b^2}{b^2 - c^2}$$

$$\phi = \arccos\frac{c}{a},\ k = \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}$$

$$a^2 M_1 + b^2 M_2 + c^2 M_3 = M_0$$

Here $F(\phi, k)$ and $E(\phi, k)$ are incomplete elliptic integrals of the first and the second kind.

Using the expressions for A we obtain, in the first approximation, the following expressions for the electromagnetic field inside the elliptical bunch of particles due to the space charge:

$$B_1 = -\frac{p}{\gamma}\rho_m M_2 x_2,\ B_2 = \frac{p}{\gamma}\rho_m M_1 x_1,\ B_3 = 0$$

$$E_1 = \rho_m M_1 x_1,\ E_2 = \rho_m M_2 x_2,\ E_3 = \rho_m M_3 x_3$$

4 PARAXIAL EQUATIONS OF MOTION

The motion of the elliptical bunch of charged particles in arbitrary space-time is described by the following paraxial equation:

$$y' = Py \qquad (3)$$

where:

$$y_1 = x_1,\ y_2 = x_2,\ y_3 = x_3,\ y_4 = \frac{p}{p_0}x'_1,\ y_5 = \frac{p}{p_0}x'_2,$$

$$y_6 = \frac{p_0^3}{p^3}x'_3,\ P = \begin{pmatrix} 0 & C \\ F & D \end{pmatrix}$$

$$C_{ik} = 0,\ i \ne k,\ C_{11} = C_{22} = \frac{p_0}{p},\ C_{33} = \frac{p^3}{p_0^3}$$

$$F_{1k} = \frac{1}{p_0}\left(-\nabla_k B_{m2} + \frac{\gamma}{p}\nabla_k E_{m1} + \delta(k,1)\frac{\rho_m}{p\gamma}M_1\right)$$

$$F_{2k} = \frac{1}{p_0}\left(\nabla_k B_{m1} + \frac{\gamma}{p}\nabla_k E_{m2} + \delta(k,2)\frac{\rho_m}{p\gamma}M_2\right) \qquad (4)$$

$$F_{3k} = \frac{p_0^3}{p^5\gamma}\left(\nabla_k E_{m3} + \delta(k,3)\rho_m M_3\right)$$

$$D_{kk} = 0,\ k = 1,\ 2,\ 3;\ D_{12} = -D_{21} = \frac{B_{m3}}{p}$$

$$D_{31} = \frac{p_0^4}{\gamma p^6}E_{m1},\ D_{13} = -\frac{\gamma p^2}{p_0^4}E_{m1},\ D_{32} = \frac{p_0^4}{\gamma p^6}E_{m2}$$

$$D_{23} = -\frac{\gamma p^2}{p_0^4}E_{m2}$$

$$\nabla_k B_{mj} = \frac{\partial B_j}{\partial x_k}\Big|_{x=0},\ \nabla_k E_{mj} = \frac{\partial E_j}{\partial x_k}\Big|_{x=0}$$

Here B and E are external electromagnetic fields and p_0 is the initial value of p at $z_{m4} = z_{m40}$. The *Kronecker delta* is defined as $\delta(i,j) = 1$ if $i = j$ and $\delta(i,j) = 0$ if $i \neq j$. Assuming that the charge of the bunch is given by:

$$Q = \int_{\Omega(x)} \rho^* dx_1 dx_2 dx_3 = \frac{4}{3}\pi a_1 a_2 a_3 \rho_m^*$$

we obtain:

$$\rho_m^* = \frac{I}{\frac{4}{3}\pi a_1 a_2 a_3 \nu} = \frac{I}{\frac{4}{3}\pi abc\nu}$$

where ν is the frequency of the beam current impulses.

5 The recursive σ—method of solution

The solution of the matrix equation of motion eq. (3) is written in terms of the matrizant in the form:

$$y = Ry_0, \ R_0 = I_6$$

where R is a 6x6 matrix, I_6 is a 6x6 unit matrix and R_0 and y_0 are the initial values of R and y, resp. A continuous, generalized, analogue of the Gauss brackets or the method of shuttle integrals, refs. [2] and [9], can be used to calculate the matrizant for an arbitrary coefficient matrix $P(z_{m4})$ with rigorous conservation of the phase volume of the beam at each stage of the calculation.

We are considering the bunch of particles which is the 6-dimensional ellipsoid in phase space. It is convenient to use the envelope matrix σ, where:

$$\sigma = R\sigma_0\tilde{R}$$

Here R is the matrizant and σ is the matrix characterizing the shape of the initial 6-dimensional ellipsoid in the phase space. We note that:

$$\sigma_{11} = a_1^2(z_{m4}), \ \sigma_{22} = a_2^2(z_{m4}), \ \sigma_{33} = a_3^2(z_{m4})$$

$$\sigma_{44} = \left(\frac{p}{p_0}x'_{1\,\max}\right)^2, \ \sigma_{55} = \left(\frac{p}{p_0}x'_{2\,\max}\right)^2$$

$$\sigma_{66} = \left(\frac{p_0^3}{p^3}x'_{3\,\max}\right)^2$$

where $x'_{j\,\max}$ is the maximum value of x'_j in the phase set with an ellipsoid boundary.

The matrix σ satisfies the differential equation:

$$\sigma' = P\sigma + \sigma\tilde{P}$$

where the coefficient matrix P is described by the eqs. (4). It is to be noted that the matrix P depends of the elements σ_{11}, σ_{22} and σ_{33}.

6 Summary

We have considered the motion of the 3-dimensional elliptical bunch of particles in an arbitrary, external, electromagnetic field, taking into account the effect of space-charge. The space-charge effect is important for a high beam current I, for a bunch of small volume and for beam current impulses of low frequency ν. In the first approximation we have found the electromagnetic field inside the bunch due to the space charge. This is given in eq. (2). A new recursive technique has been proposed for the solution of the non-linear equations of motion in the first approximation, given by eq. (3) where, in each step of the numerical integration, the phase volume of the beam is strictly conserved.

Acknowledgement

This work was supported by the Swedish-Soviet Research Cooperation, established between the former USSR Academy of Sciences and the Royal Swedish Academy of Sciences.

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