Emittance Growth in Displaced, Space-Charge-Dominated Beams with Energy Spread*

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Abstract

Conversion of transverse energy associated with the coherent motion of displaced beams into thermal energy, and thus emittance growth, has been predicted theoretically by a number of authors. Here, we show, using 2-D particle-in-cell simulations, that emittance growth is inhibited for tune depressed beams, if the energy spread of the beam is not too large. Further, using a uniform density model to calculate the space charge field of the beam, we numerically determine the criteria for emittance growth as a function of tune depression, energy spread, and beam displacement over a wide range of parameters. A theoretical interpretation of our results is presented.

I. INTRODUCTION

In an inertial fusion energy reactor, driven by a heavy ion accelerator, the ion beam must be focused down to a small (2-3 mm) spot at the target. The normalized emittance, (which is a measure of the transverse phase area occupied by the beam) must be sufficiently small in order to meet the spot size requirement. An understanding of emittance growth is thus of paramount importance to ensure optimum performance of a driver. One source of emittance growth is the conversion of the energy associated with coherent oscillation of the beam centroid into thermal energy of the beam. These oscillations may arise from initially non-aligned beams or accumulated from small misalignments in the focusing quadrupoles, for the case of a linear accelerator. In addition, centroid oscillations may arise from errors in the field strength in bending magnets, or voltage errors in the acceleration modules in accelerators in which bends are present.

It is well known, that a beam with a KV distribution in the transverse direction [1] and monoenergetic in the longitudinal direction, and subject to strictly linear focusing forces will undergo no emittance growth, even if the beam is initially displaced. It is also well known that for emittance dominated beams (in which space charge is negligible), a small spread in the longitudinal velocity gives rise to a spread in the betatron frequency of individual particles, and this spread mixes the phase of the oscillating particles, removing the coherent oscillation of the centroid as a whole and increasing the effective transverse area of phase space occupied by the beam. For emittance dominated beams, the smaller the energy spread, the smaller the rate at which the emittance grows to its final phase-mixed value. However, the asymptotic value of the emittance is nearly independent of energy spread, and can be estimated by using the conservation of transverse energy.

In this paper, we examine the effect of energy spread on beams with finite space-charge-depressions in linear focusing channels and find behavior which is qualitatively different from the emittance dominated case. In particular, we find that for large initial centroid displacements there exists a threshold in energy spread, below which, no significant emittance growth occurs, and above which, the beam approaches the asymptotic phase mixed state. For small beam displacements, the transition is not sharp but the same general trend occurs.

II. 2-D PARTICLE-IN-CELL RESULTS

The effect was first observed numerically, using the 2-D slice code SHIFTXY [2]. Here x and y are the transverse coordinates, and prime indicates derivative with repect to z the longitudinal coordinate. The beam is assumed to be displaced $x_c = \langle x \rangle$ in the x direction only, where $\langle\ \rangle$ indicates average over particles in a slice. The beam widths in both x and y directions were matched for the emittance and current of the beam.

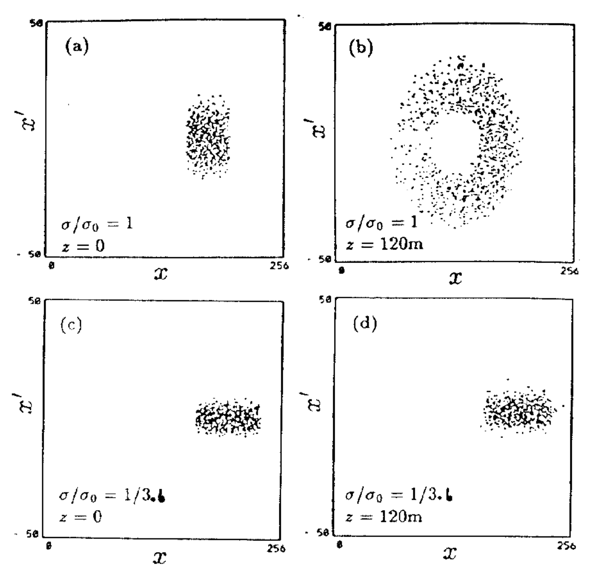

Fig. 1. x-x' phase space plots for two beams as indicated.

Fig. 1 illustrate the effects. In fig. 1a, the $x - x'$ phase space is shown for an undepressed beam ($\sigma/\sigma_0 = 1$), initially at $z = 0$ and in fig. 1b, eight betatron periods later (z =120 m). Here σ_0 and σ are the undepressed

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and depressed phase advances, respectively. The beam had fractional momentum spread of halfwidth $\Delta = 0.01$, and the initial centroid displacement $x_{c0} = 2a$, where a is the initial beam radius in x. The beam fills a ring in $x - x'$ space of nearly uniform density, as different particles with different betatron frequencies, phase mix, damping the coherent centroid oscillation. In figure 1c and 1d, the initial and final phase spaces are shown for $\sigma/\sigma_0 = 1/3.6$ with the same momentum spread. Note the transverse area is nearly unchanged from the initial area. This reduction in emittance growth (in this case by two orders of magnitude) is an unexpected benefit of space-charge dominated beams.

III. MODEL RESULTS

In order to obtain an understanding of this behavior we may adopt model equations of motion which simplify the physics of the space-charge force. We assume that the equations of motion of for each ion are given by,

$$x_i'' = \left(-k_{\beta 0}^2 x_i + k_{sx}^2(x_i - x_c)\right)/(1+\delta_i)^2. \qquad (1)$$

$$y_i'' = \left(-k_{\beta 0}^2 y_i + k_{sy}^2(y_i - y_c)\right)/(1+\delta_i)^2. \qquad (2)$$

Here, subscript i indicates the coordinates of the ith particle in a particular slice in z, which is traveling in the $+z$ direction; $k_{\beta 0} \equiv \sigma_0/(2L)$ represents magnetic FODO focusing in the smooth approximation, and L is the lattice half-period; $\delta_i = \delta p_i/p$ is the fractional difference between the momentum of the ith particle and the average momentum; $K \equiv 2qI/(\beta^3 A I_o)$ is the perveance, where q is the charge state of the ions, A is the atomic mass of the ions, β is the average longitudinal ion velocity in units of c, $I_o \equiv m_p c^3/e$ is the proton Alfven current ($\cong$31 MA). Also,

$$k_{sx}^2 \equiv K/[2(\Delta x^2 + (\Delta x^2 \Delta y^2)^{1/2})],$$
$$k_{sy}^2 \equiv K/[2(\Delta y^2 + (\Delta x^2 \Delta y^2)^{1/2})]. \qquad (3)$$

Here $\Delta x^2 = \langle x^2\rangle - \langle x\rangle^2$, and $\Delta y^2 = \langle y^2\rangle - \langle y\rangle^2$. Eqs. (1) and (2) represent in an approximate way, the effects of magnetic focusing and space charge, and the $(1+\delta_i)^{-2}$ factor indicates the velocity dependence of these forces when z is the independent variable. The physical approximations that have been made include the following: (1) Focusing is smooth and not a function of z (k_{β_0} is constant). (2) In eqs. (1) and (2) terms through first order in the small quantities $k_{\beta 0}x_i$, $k_{\beta 0}y_i$ have been kept. Non-linearities arise through the dependence of k_{sx} and k_{sy} on Δx^2 and Δy^2 and phase mixing can occur because of the $(1+\delta_i)^{-2}$ factors. (3) Space charge forces depend only on lowest order moments. (We have used the electrostatic potential of a uniform density elliptical beam, allowing variation with z of the centroid position and semi-major axes.) (4) The beam is not undergoing acceleration: (p, β, and δp_i are constants). (5) The beam is non-relativistic: ($\beta << 1$).

It has been shown [3,4], that when the factor $(1+\delta_i)^{-2}$ is set to unity in equations (1) and (2), a transverse energy H may be defined which is constant in z:

$$2H = k_{\beta 0}^2(\Delta x^2 + \Delta y^2) + \Delta x'^2 + \Delta y'^2$$
$$-K\ln\left((\Delta x^2)^{1/2} + (\Delta y^2)^{1/2}\right)$$
$$+k_{\beta 0}^2 x_c^2 + k_{\beta 0}^2 y_c^2 + x_c'^2 + y_c'^2 \qquad (4)$$

When the factor $(1+\delta_i)^{-2}$ is included, H is no longer precisely constant, but the fluctuations in H are small when Δ is small.

Phase mixing can lead to a final state in which the beam centroid has decayed to zero, at the expense of a larger phase space area. Two final states of the beam are of interest. When the x and y equations of motion are sufficiently uncoupled (as for example when $\sigma \cong \sigma_0$) the beam width and displacement in the y direction are unaffected by the initial displacement in x. The final state of the beam evolves to $x_{cf} = y_{cf} = x'_{cf} = y'_{cf} = 0$ and $\Delta y_f^2 = \Delta y_0^2$, where subscripts 0 and f stand for initial and final, respectively. Conservation of the transverse energy, yields:

$$2H_0 = k_{\beta 0}^2(\Delta x_f^2 + \Delta y_0^2) + k_{sxf}^2\Delta x_f^2 + k_{syf}^2\Delta y_0^2$$
$$-K\ln[(\Delta x_f^2)^{1/2} + (\Delta y_0^2)^{1/2}] \qquad (5)$$

Here $2H_0 = 2(k_{\beta 0}^2 + k_{sx0}^2)\Delta x_0^2 - K\ln[2(\Delta x_0^2)^{1/2}] + k_{\beta 0}^2 x_{c0}^2$. Equation (5) may be solved (numerically) for Δx_f^2. The root mean square width in x' is given by $\Delta x_f'^2 = k_{sxf}^2\Delta x_f^2$ and similarly the width in y' is given by $\Delta y_f'^2 = k_{syf}^2\Delta y_f^2$ (ref. 4). The final x emittance is given by $\epsilon_{xf} = 4(\Delta x_f^2\Delta x_f'^2)^{1/2}$ and similarly in y, $\epsilon_{yf} = 4(\Delta y_f^2\Delta y_f'^2)^{1/2}$.

The second final state of interest is the case where the x and y equations of motion are sufficiently coupled so that the final state of the beam is the same in both x and y. In that case, $\Delta x_f^2 = \Delta y_f^2$ and

$$2H_0 = 2(k_{\beta 0}^2 + k_{sxf}^2)\Delta x_f^2 - K\ln[2(\Delta x_f^2)^{1/2}] \qquad (6)$$

Again, eq. (6) may be solved numerically for Δx_f^2, and then $\Delta x_f'^2$, $\Delta y_f'^2$, ϵ_{xf}, and ϵ_{yf} may be calculated as in the first case above.

We have integrated the model eqs. (1) and (2), over a distance of 33 betatron periods for an ensemble of particles, for a variety of ratios of beam radius $a = 2(\Delta x_0^2)^{1/2}$ to initial beam displacements x_c, space charge depressions σ/σ_0, and halfwidth Δ of distribution in δ. The difference between final and initial x-emittance is shown in figures 2 and 3 for $a/x_c = 1$. Note the sharp threshold for emittance growth. The threshold may be heuristically derived by a consideration of equation (1). In the case of no energy spread the centroid oscillates sinusoidally, $x_c = x_{c0}\cos k_{\beta_0}z$, and the beam widths are constant $\Delta x^2 = \Delta y^2 = \Delta x_0^2$. If we integrate eq. 1, assuming these values for x_c, Δx^2 and Δy^2 respectively, we find the solution to eq. (1):

$$x_i = \left(x_{i0} - \frac{k_{s\delta}^2 x_{c0}}{k_{\beta 0}^2 - k^2}\right)\cos kz + \frac{x'_{i0}\sin kz}{k} + \frac{k_{s\delta}^2 x_{c0}\cos k_{\beta_0}z}{k_{\beta 0}^2 - k^2} \qquad (7)$$

Here $k^2 \equiv (k_{\beta 0}^2 - k_{sx}^2)/(1+\delta_i)^2 \equiv (\sigma/2L)^2/(1+\delta_i)^2$. Also, $k_{s\delta}^2 \equiv k_{sx}^2/(1+\delta_i)^2$. Note that because of the energy spread, the assumption that Δx^2 is constant breaks down. A necessary condition for eq. (7) to be nearly self-consistent is that the difference between the assumed x_c and a particle position x at the edge of the distribution in δ_i be much less than some fraction f of a beam radius. This condition leads to the equation:

$$\left(\frac{k_{i\delta}^2}{k_{\beta 0}^2-k^2}-1\right)x_{c0}\le fa \qquad (8)$$

Eq. 8 defines a critical δ_i below which (*i.e.* for large negative values) the equations of motion become very nonlinear, but above which the linear solution (eq. 7) remains approximately valid. When we substitute the half-width Δ of a uniform distribution in δ_i for $-\delta_i$ into equation (8) that threshold may be written

$$\Delta = 1-\left(\frac{1+(\sigma^2/\sigma_0^2)fa/x_c}{1+fa/x_c}\right)^{1/2}. \qquad (9)$$

This threshold closely matches the threshold found in the numerical integration of eqs. 1 and 2 for emittance growth, when the value of $f = 0.6 + 0.5(a/x_{c0})$. This formula for f was found by fitting to the numerical plots. When the threshold is exceeded the emittance growth is given closely by that calculated from eq. (5) (see fig. 4). This threshold is sharp when $a/x_{c0} < 1$, but the simulations show that the threshold is much broader when $a/x_{c0} >> 1$. In that case, the asymptotic emittance is close to the initial matched emittance, and the transition from zero emittance growth to asymptotic emittance growth is gradual, so that the precise value of f is not well defined.

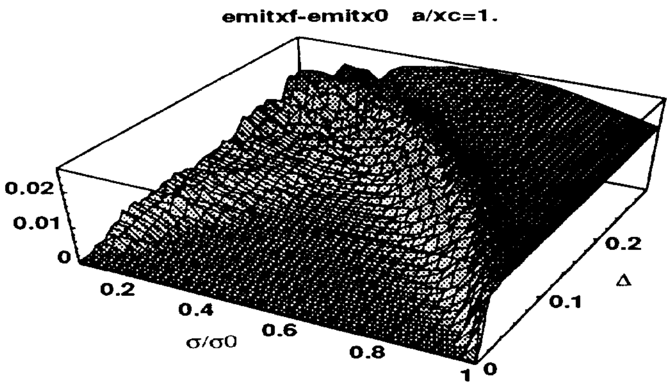

Figure 2. Model equation results of $\epsilon_{xf} - \epsilon_{x0}$ for $a/x_{c0} = 1.0$.

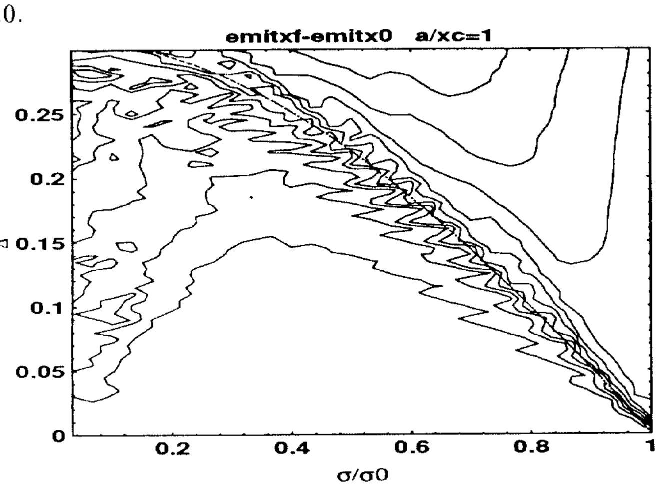

Figure 3. Contour plot of figure 2, ($\epsilon_{xf} - \epsilon_{x0}$ for $a/x_{c0} = 1.0$). Dotted line is the threshold, (eq. 9).

IV. DISCUSSION AND CONCLUSION

Although, use of eqs. 1 and 2 simplify the actual forces experienced by the particles, threshold predicted by eq. 9 is qualitatively obtained in the particle-in-cell (PIC) results, and the final emittance predicted by eq. 5 is also approximately obtained in the PIC results, although

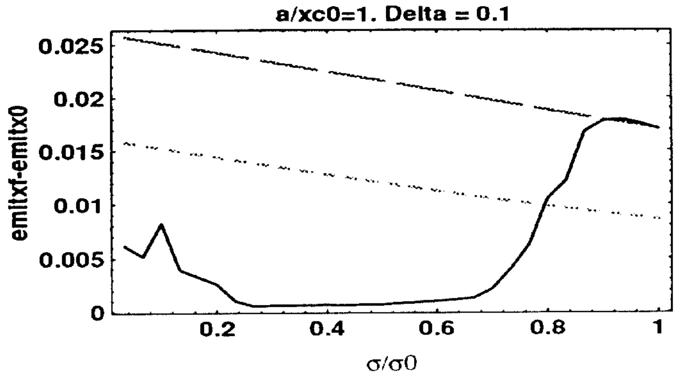

Figure 4. Model equation results of$\epsilon_{xf} - \epsilon_{x0}$ vs. σ/σ_0 for $\Delta = 0.10$. Dashed line is asymptotic emittance found using eq. 5. Short dashed line is found using eq. 6.

additional phase mixing to the final state predicted by eq. 6 is seen for smaller σ/σ_0. The result that space-charge depression reduces chromatic aberration in an alternating gradient lattice was observed by Lee [5] and is physically related to the reduction of phase mixing for displaced beams. The coherence of centroid oscillations of highly depressed beams over many betatron periods was unexpected and was not predicted in previous studies of emittance growth from displaced beams [3,6]. Also, as indicated above, sufficiently large longitudinal velocity spreads will give rise to non-linearities which may become fully phase mixed. These results will have applications in determining error tolerances in possible drivers for heavy-ion inertial fusion.

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