SPACE-CHARGE-INDUCED EMITTANCE GROWTH IN AN ELLIPTICAL CHARGED PARTICLE BEAM WITH A PARABOLIC DENSITY DISTRIBUTION

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Abstract

We present a simple analytical model of emittance growth caused by nonlinear space-charge forces using a laminar, parabolic-density beam. The model allows us to explain a surprising and important result that space-charge-induced emittance growth is larger in the plane with the larger semi-axis.

I. INTRODUCTION

We consider a model, which describes the process in a short drift space, in which the particles experience a space-charge impulse, but do not propagate far enough for their positions to change appreciably. Thus the spatial distribution is assumed to remain unchanged as the beam propagates. We consider a continuous beam of elliptical cross section, propagating in the z direction. The horizontal semiaxis is a and the vertical semiaxis is b. The beam current is given by $I = qN_l v$, where q is the charge per particle, v is the beam velocity, and N_l is the number of particles per unit length..

For an assumed particle density n(x,y), Poisson's equation provides the basis for obtaining the space-charge field components (assuming nonrelativistic beams and ignoring magnetic forces). The solution for the electric-field components from a distribution with elliptical symmetry has been given by Sacherer[1]. If the density function has the form

$$n(x,y) = n\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) ,$$

the x and y components of the field are given by

$$E_x = \frac{qabx}{2\varepsilon_0}\int_0^\infty ds\; n\left(\frac{x^2}{a^2+s}+\frac{y^2}{b^2+s}\right)\left(a^2+s\right)^{-3/2}\left(b^2+s\right)^{-1/2} ,$$

and

$$E_y = \frac{qaby}{2\varepsilon_0}\int_0^\infty ds\; n\left(\frac{x^2}{a^2+s}+\frac{y^2}{b^2+s}\right)\left(a^2+s\right)^{-1/2}\left(b^2+s\right)^{-3/2} .$$

From the field components the transverse momentum impulse can be calculated as a function of position. This results in a new phase-space distribution (changed in momentum space), and a new rms emittance. For the x plane the change in the momentum component is

$$\Delta p_x = \frac{qE_xL}{v} ,$$

Work supported in part by Los Alamos National Laboratory Institutional Supporting Researches under the auspices of the United States Department of Energy.

where L is the length of the drift space. The impulse can also be expressed as a change in the divergence angle, given nonrelativistically in the paraxial approximation by

$$\Delta x' = \frac{qE_xL}{mv^2} ,$$

where m is the mass of the beam particles.

Suppose the initial beam is idealized by assuming the particle distribution is described by a straight line in phase space, given by

$$x_0' = A\frac{x_0}{a} .$$

The rms emittance is defined as

$$\varepsilon = \left[\overline{x^2}\,\overline{x'^2} - \left[\overline{xx'}\right]^2\right]^{1/2} .$$

If the final second moments of the particle distribution can be evaluated from the expression for x and x′, the final rms emittance can be obtained. The positions are assumed to remain fixed so

$$x = x_0 .$$

The final divergence is

$$x' = A\frac{x_0}{a} + \frac{qE_xL}{mv^2} .$$

To go further, we need to specify the transverse spatial-density distribution. The uniform density beam, which gives a linear defocusing space-charge force, leads to defocusing effects with no rms emittance growth[2]. The parabolic-density beam leads to a nonlinear space-charge force, and growth of the rms emittance.

II. PARABOLIC DENSITY ELLIPTICAL BEAM

Suppose that the density is parabolic, given by

$$n(x,y) = \frac{2N_l}{\pi ab}\left[1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right] ,$$

within the boundary of the ellipse defined by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 .$$

Outside the ellipse, the density is assumed to equal zero. In this section we will treat the problem in the x plane; the y-plane result can be obtained in a similar way. The x component of the electric field is given by

$$E_x = \frac{qN_1 x}{\pi\varepsilon_0} I_x \ ,$$

where

$$I_x = \int_0^\infty ds\left(1 - \frac{x^2}{a^2+s} - \frac{y^2}{b^2+s}\right)\left(a^2+s\right)^{-3/2}\left(b^2+s\right)^{-1/2} .$$

The integral I_x can be evaluated to obtain the expression for the field.[2]

The x component of the space-charge electric field is

$$E_x = \frac{2qN_1}{\pi\varepsilon_0}\left[\frac{x}{a(a+b)} - x^3\left[\frac{(2a+b)}{3a^3(a+b)^2}\right] - xy^2\left[\frac{1}{ab(a+b)^2}\right]\right] .$$

By symmetry, the y component is

$$E_y = \frac{2qN_1}{\pi\varepsilon_0}\left[\frac{y}{b(a+b)} - y^3\left[\frac{(2b+a)}{3b^3(a+b)^2}\right] - yx^2\left[\frac{1}{ab(a+b)^2}\right]\right] .$$

The field components contain a linear term, a cubic term, and a coupling term that depends on both coordinates.For the uniform density case, we have[2]

$$E_x(a,0) = E_y(0,b) \ .$$

For the parabolic density this result is not true. Instead, we have

$$E_x(a,0) = \frac{2qN_1}{\pi\varepsilon_0(a+b)}\left[1 - \frac{2+b/a}{3(1+b/a)}\right],$$

and

$$Ey(0,b) = \frac{2qN_1}{\pi\varepsilon_0(a+b)}\left[1 - \frac{2b/a+1}{3(1+b/a)}\right].$$

Curves of $E_x(a,0)$ and $E_y(0,b)$ in relative units versus the ratio a/b are shown in Fig. 1. We see from the curves that, although these fields are not exactly equal except when the beam is round, they are within about 20% over a large range of a/b; thus these electric field components are generally comparable in magnitude. We will see later that this approximate symmetry affects the scaling of emittance growth with beam size.

In the round beam limit one obtains

$$E_x = \frac{qN_1}{\pi\varepsilon_0 b^2}\left[x - \frac{x^3}{2b^2} - \frac{xy^2}{2b^2}\right],$$

and the round-beam expression for E_y is

$$E_y = \frac{qN_1}{\pi\varepsilon_0 b^2}\left[y - \frac{y^3}{2b^2} - \frac{yx^2}{2b^2}\right],$$

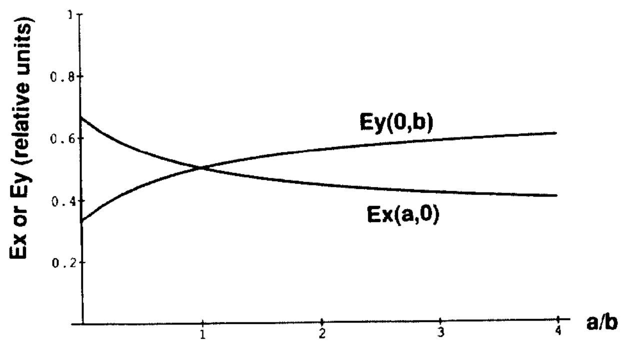

Figure 1. Ex(a,0) and Ey(0,b) in relative units versus ellipse aspect-ratio a/b for the parabolic-density distribution.

Now consider the effect of the coupling term for E_x. For any x value, the y value ranges from $-b(1-x^2/a^2)^{1/2}$ to $+b(1-x^2/a^2)^{1/2}$. The minimum value of E_x corresponds to $y^2=b^2(1-x^2/a^2)$, and the maximum value of E_x occurs when $y^2=0$. These limiting curves are plotted versus x/a in Fig. 2. We see that the coupling of the y coordinate into the x field component results in a range of E_x values for any x coordinate.

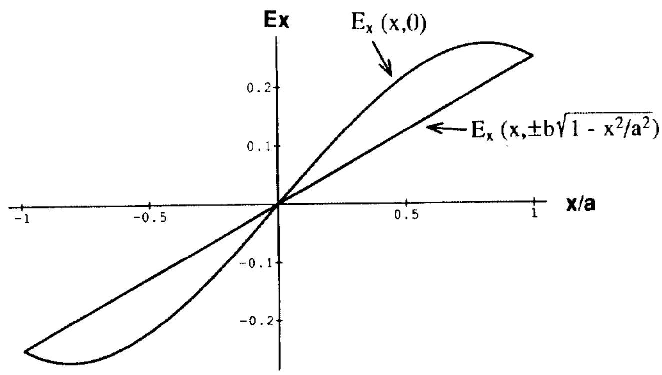

Figure 2. Ex versus x/a for a parabolic-density beam. The curves corresponding to y=0 and $\pm b\sqrt{1-x^2/a^2}$ define the maximum and minimum values of the space-charge electric field.

The space-charge kick in the x plane produces a divergence change, which depends on both x and y. The final divergence can be written, using the expression for the field. The rms emittance, calculated from the second moments of the distribution, is[2]

$$\varepsilon_x = KL\frac{a}{b}\sqrt{\frac{1}{432}\frac{\left(\frac{2a}{b}+1\right)^2}{\left\{1+\frac{a}{b}\right\}^4} + \frac{7}{720}\frac{1}{\left\{1+\frac{a}{b}\right\}^4} - \frac{1}{360}\frac{\left(\frac{2a}{b}+1\right)}{\left\{1+\frac{a}{b}\right\}^4}} .$$

The emittance expression for the y plane is obtained by interchanging a and b. The first term corresponds to the filamentation effect caused by the cubic term in the field, and represents the rms-emittance increase from the resulting curvature in phase space. The second term comes from the coupling, i.e. the dependence of the x component of the field on the y coordinate, which produces the spreading of the initial filament. The last term is a cross term between the filamentation term and the coupling term.

For the round beam, where a = b, this reduces to

$$\varepsilon_x = \frac{KL}{\sqrt{720}} = \frac{KL}{26.833},$$

which is independent of beam size. The initial and final phase-space configurations are shown in Fig. 3. The nonlinearity has changed the initial straight line into a distribution with an average curvature, which represents the beginning of the filamentation process where the phase-space distribution begins to bend. Of course as time increases the beam distribution also changes and for longer drifts, this must also be taken into account. The coupling has produced the spreading effect, which changes the initial line in phase space into a finite area. The spreading is zero at the origin and at the edge of the beam.

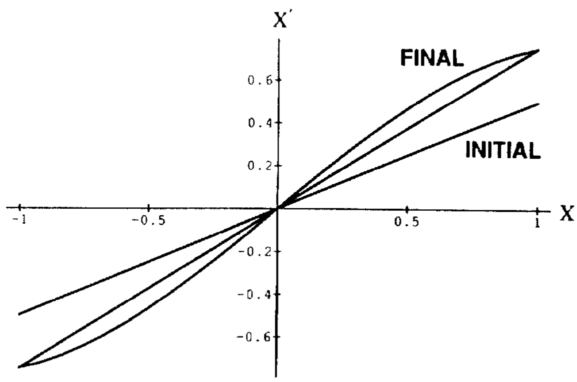

Figure 3. Effect of space charge from a parabolic density on an initial zero-emittance beam. The initial and final phase-space distributions are shown.

III. DEPENDENCE OF THE RMS EMITTANCE ON ELLIPSE ASPECT RATIO

In Fig. 4 we show the plot of the x and y final emittances versus the semiaxis ratio a/b. The final x emittance increases and the y emittance decreases with increasing a/b. That the emittance growth increases as the semiaxis length increases may seem surprising for an effect that arises from space-charge force, which increases as the beam size becomes smaller,rather than larger. The explanation is that the relative field components, and therefore the relative divergence kicks in x and y, are insensitive to the semiaxis sizes, as we saw earlier (see Fig. 1). However, emittance is an area in phase space, and is essentially a product of the divergence spread times the spatial extent of the beam in that plane. Therefore, if the divergence kicks are comparable in the two planes, the emittance growth is larger in the plane that has the larger semiaxis length.

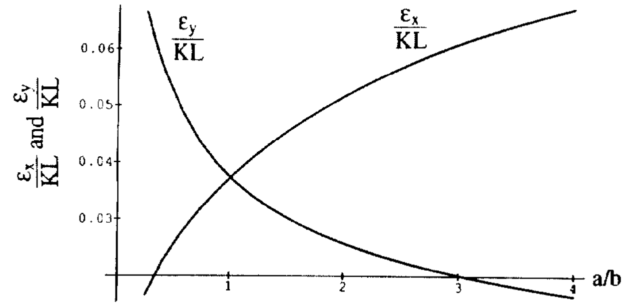

Figure 4. Final rms emittance values versus ellipse-aspect ratio a/b for a beam with parabolic density.

A closer examination of the effects of the different terms in the expression for ε_x^2 can be made by plotting each of these three terms versus the semiaxis ratio a/b, as shown in Fig. 5. The filamentation term dominates for a/b > 1. The coupling or spreading term dominates in the region given approximately by a/b < 0.5. The third or cross term is never the dominant term. One sees that the coupling term does decrease with increasing a/b when a/b > 1, but this occurs when the dominant term is the filamentation term

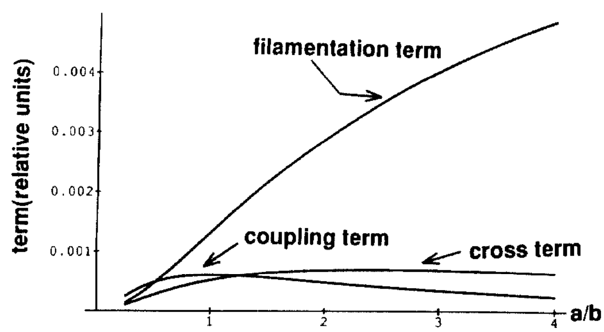

Figure 5. The three terms in the expression for ε_x^2 versus ellipse aspect ratio a/b.

IV. ACKNOWLEDGMENTS

We acknowledge some valuable discussions with our colleagues M. Weiss and E. Tanke at CERN.

V. REFERENCES

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[2] T. P. Wangler, P. Lapostolle, and A. Lombardi, "A Model of Non-Linear Space-Charge Forces in a Charged Particle Beam", CERN report CERN/PS 93-11 (HI), March 1993, and Los Alamos AT-Division Technical Note No. AT-1:92-416, December 15, 1993.