Proton-Proton Scattering Contribution to Emittance Growth

T. Garavaglia
Superconducting Super Collider Laboratory*
2550 Beckleymeade Ave., Dallas, TX 75237 USA

Abstract

Proton-proton scattering contributes to the emittance growth of the SSC. A formulation is given and used to estimate the mean scattering angle, which is used to determine the contribution to SSC emittance growth resulting from elastic *pp* scattering. The method is based upon Lorentz invariants, and it permits the determination of the cross-section for scattering in the center of mass (c.m.) system, as well as scattering from a fixed target (f.t.). Also an example is given for the case of electromagnetic *pp* scattering, which results from single virtual photon exchange.

I. INTRODUCTION

The contribution from *pp* elastic scattering to transverse emittance growth is determined for the SSC. Elastically scattered protons with a small scattering angle will remain within the Collider proton beam. These scattered particles contribute to the growth of the beam's transverse emittance. Numerical results for emittance growth resulting from *pp* scattering and other sources are given in [1]. In this paper, a summary is given of the analytical methods that are used to determine the mean scattering angle resulting from *pp* elastic scattering. Lorentz invariants and cross-sections are defined in Appendix A.

II. PROTON-PROTON ELASTIC SCATTERING

The contribution to transverse emittance growth, for one degree of freedom, resulting from *pp* elastic scattering is given by

$$\left(\frac{d\epsilon_x}{dt}\right) = (\beta^*_{1x}\mathcal{L}_1 + \beta^*_{2x}\mathcal{L}_2)\sigma_{el} < \theta_x^2 > /(MN_B), \quad (2.1)$$

where β^*_i and $\mathcal{L}_i$ are, respectively, the beta function and the luminosity at the i^{th} interaction point. In this expression N_B is the number of protons per bunch, M is the number of bunches, σ_{el} is the *pp* elastic scattering cross-section, and $\sqrt{< \theta_x^2 >}$ is the rms value of the *pp* elastic scattering angle in the center of mass system, which is projected onto the transverse x-direction. A similar expression occurs for the transverse y-direction. The mean scattering angle can be estimated from the differential elastic scattering cross-section

$$\left(\frac{d\sigma_{el}}{d\Omega}\right)_{c.m.} = \frac{S}{4\pi}\left(\frac{d\sigma_{el}}{dT}\right)_{c.m.}, \quad (2.2)$$

where S and T are Lorentz invariants. These invariants, defined in (A1), are $S \approx 2E_{c.m.}$, and $T \approx -S\sin^2(\theta/2) \approx -S(\theta_x^2 + \theta_y^2)/4$. The invariant differential cross-section for *pp* elastic scattering is [2]

$$\frac{d\sigma_{el}}{dT} \approx \frac{\sigma_T^2}{16\pi}(1+\rho^2)e^{bT}, \quad (2.3)$$

where σ_T is the total *pp* cross-section, b is the slope parameter, and $\rho \approx 0$ is the ratio of the real part to the imaginary part of the scattering amplitude. Assuming that the slope parameter is a constant, one can integrate the differential cross-section to obtain $b \approx \sigma_T^2/16\pi\sigma_{el}$. With the approximation $\sigma_{el} \approx (1/4)\sigma_T$, one finds $b \approx \sigma_T/4\pi$.

Using the approximations above, the differential cross-section in the c.m. system becomes

$$\left(\frac{d\sigma_{el}}{d\Omega}\right)_{c.m.} \approx \frac{S\sigma_T^2}{64\pi^2}e^{\left(\frac{\theta_x^2}{2\sigma^2_{\theta_x}}+\frac{\theta_y^2}{2\sigma^2_{\theta_x}}\right)}, \quad (2.4)$$

where

$$\sigma_{\theta_x} = \sqrt{< \theta_x^2 >} = (bS/2)^{-1/2}. \quad (2.5)$$

This is the expression to be substituted into (2.1) to find $d\epsilon_x/dt$. For colliding proton beams with $\sqrt{S} = 40$ TeV and $\sigma_T \approx 130$ mb, one finds $b \approx 26.6\ \text{GeV}^{-2}$ and

$$\sigma_{\theta_x} = \sqrt{< \theta_x^2 >} = 6.9\,\mu\,\text{rad}. \quad (2.6)$$

In the above, the value of the total *pp* scattering cross-section is determined from

$$\sigma_T = 38.5 + 1.33 ln^2\left(\frac{\sqrt{S}}{10\,\text{GeV}}\right), \quad (2.7)$$

which is obtained from cosmic ray data [3]. Theoretical models giving values for the total and elastic *pp* cross-sections can be found in [4].

Using the above techniques, one can estimate the rms scattering angle $\sqrt{< \tilde{\theta}^2 >}$ for the scattering of a proton in a 20 TeV beam from a fixed proton. The scattering angle in the c.m. system for high energy *pp* scattering is found from

$$\cos(\theta) \approx \frac{2T}{S} + 1. \quad (2.8)$$

*Operated by the Universities Research Association, Inc., for the U.S. Department of Energy under Contract No. DE-AC35-89ER40486.

The scattering angle for a proton of energy $\omega = S/2m$ from a proton at rest is found from

$$\cos(\tilde{\theta}) \approx 1 - \frac{2\sin^2(\theta/2)}{(S/m^2)} - \frac{4}{(S/m^2)^2}. \qquad (2.9)$$

Using $\cos\tilde{\theta} \approx 1 - (1/2)\sin^2\tilde{\theta}$, the fixed target scattering angle is related to the c.m. scattering angle θ through

$$\tilde{\theta} \approx \frac{\sqrt{2}}{(\omega/m)}\sqrt{\left(\frac{\theta^2(\omega/m)}{4} + 1\right)}. \qquad (2.10)$$

For the scattering of a 20 TeV proton from a proton at rest, the rms scattering angle in the c.m. system is found from (2.5),with $\sqrt{S} = 193$ GeV, $\sigma_T = 50.2$ mb and $b \approx 10.3\,\mathrm{GeV}^{-2}$, to be $\sqrt{<\theta^2>} \approx 3.2$ mrad. The corresponding angle in the fixed target system is found to be $\sqrt{<\tilde{\theta}_x^2>} \approx 47\,\mu$rad.

III. PROTON-PROTON ELECTROMAGNETIC SCATTERING

In this example, pp scattering is treated as an electromagnetic event, and the scattering of two fermions of initial four-momenta a and b to a final state of four-momenta c and d results from the exchange of a single virtual photon. The system of units $\hbar = c = m = 1$ is used. Since both the initial and final states involve identical particles, these states must be antisymmetrical. The initial state $|I)$ and the final state $|F)$ are represented as

$$|I) = \frac{|ab) - |ba)}{\sqrt{2}} \quad |F) = \frac{|cd) - |dc)}{\sqrt{2}}. \qquad (3.1)$$

The matrix element for this process is

$$(F|M|I) = [(cd|M|ab) - (cd|M|ba) +$$

$$(dc|M|ba) - (dc|M|ab)]/2$$

$$= e^2[J^\mu(d,b)D_{\mu\nu}(a-c)J^\nu(c,a) - (d \leftrightarrow c)], \qquad (3.2)$$

where the fermion current is $J^\mu(c,a) = \bar{u}_c\gamma^\mu u_a$. The photon propagator is $D_{\mu\nu}(a-c) = 4\pi g_{\mu\nu}/((a-c)^2 + i\epsilon)$. We use the fermion density matrix $\rho_{ij}(a) = u_{ai}\bar{u}_{aj}$, which has the property $Tr\rho(a) = 2$.

The invariant differential cross-section for this process is

$$\frac{d\sigma_{el}}{dT} = \frac{1}{16\pi f(S,a,b)}\mathcal{M}(S,T), \qquad (3.3)$$

where $\mathcal{M}(S,T) = |(F|M|I)|^2$. For the scattering of unpolarized fermions when the polarization of the final state fermions is not observed, the initial state spin density matrices for a and b are of the form $\rho_0(a) = (\not{a} + 1)/2$. For the final states $|c)$ and $|d)$, which include a summation over the final spin states, the corresponding density matrices are multiplied by two. The invariant differential cross-section for this case is now found to be

$$\frac{d\sigma_{el}}{dT}(ab \to cd) = \frac{\pi e^4}{64S(S/4-1)}[A(S,T,U) + A(S,U,T)$$

$$-B(S,T,U) - B(S,U,T)]. \qquad (3.4)$$

The invariant functions are

$$A(S,T,U) = \frac{4}{T^2}T^{\mu\nu}(d,b)T_{\mu\nu}(c,a) \qquad (3.5)$$

$$B(S,T,U) = \frac{4}{TU}T^{\mu\nu}{}_{\mu\nu}(b,c,a,d), \qquad (3.6)$$

where

$$T^{\mu\nu}(c,a) = Tr[(\not{c}+1)\gamma^\mu(\not{a}+1)\gamma^\nu], \qquad (3.7)$$

$$T^{\mu\nu\lambda\sigma}(b,c,a,d) = Tr[\gamma^\mu(\not{b}+1)\gamma^\nu(\not{c}+1)\gamma^\lambda(\not{a}+1)\gamma^\sigma(\not{d}+1)]. \qquad (3.8)$$

Upon evaluation of the traces, the invariant functions become

$$A(S,T,U) = \frac{32}{T^2}[S^2 + U^2 + 8T - 8] \qquad (3.9)$$

$$B(S,T,U) = -\frac{32}{TU}[S^2 - 8S + 12)]. \qquad (3.10)$$

In the high energy limit when S becomes large, one finds

$$\frac{d\sigma_{el}}{dT}(ab \to cd) \approx \frac{4\pi e^4}{T^2}. \qquad (3.11)$$

The rms value of the c.m. scattering angle associated with (3.11) can be found using (2.8) in the form

$$<\cos\theta> \approx 1 - <\theta^2>/2 = 2<T/S> + 1, \qquad (3.12)$$

where

$$<T> = \int_{T_{min}}^{T_{max}} T(d\sigma_e/dT)dT/\sigma_e, \qquad (3.13)$$

and $\sigma_e = \int(d\sigma_e/dT)dT$. The rms value of the scattering angle is written in terms of the projection on the transverse direction as $\sqrt{<\theta^2>} = \sqrt{2<\theta_x^2>}$. The integration limits are found from $T \approx \theta^2/4$, where θ_{max} and θ_{min} are found from the uncertainty principal, $\Delta r\Delta\theta \approx \hbar/p$, and r_{max} and r_{min} are found from the beam size and proton radius, respectively. One finds for $\sigma_{el} < \theta^2 >$, which appears in (2.1), 1.6×10^{-40} m^2 from (2.3) and 1.8×10^{-42} m^2 from (3.11), which is smaller.

APPENDIX A: Kinematics and Cross-Sections

In this appendix, the kinematical variables and cross-sections used in the analysis are given. Particles characterized by four-momenta a and b interact elastically to yield particles characterized by four-momentum c and d. For this process, energy-momentum conservation is represented as $a+b = c+d$, where a typical four-vector is represented as $a = (a^0, \mathbf{a})$, such that $a^2 = (a^0)^2 - \mathbf{a}.\mathbf{a} = m_a^2$. The interaction channels are defined according to the Lorentz invariants

$$S = (a+b)^2,\ T = (a-c)^2,\ U = (a-d)^2, \qquad (A1)$$

which satisfy $(S+T+U) = a^2 + b^2 + c^2 + d^2$.

In the c.m. system, one finds the invariant expressions for energy, momentum, and scattering angle

$$\mathcal{E}_a = \mathcal{E}(S,a,b) = (S + a^2 - b^2)/4(S/4)^{1/2}, \qquad (A2)$$

$$\mathcal{E}_b = \mathcal{E}(S,b,a), \mathcal{E}_c = \mathcal{E}(S,c,d), \mathcal{E}_d = \mathcal{E}(S,d,c),$$

$$|\mathbf{a}| = |\mathbf{b}| = [\frac{f(S,a,b)}{4S}]^{1/2},\ |\mathbf{c}| = |\mathbf{d}| = [\frac{f(S,c,d)}{4S}]^{1/2}, \qquad (A3)$$

and

$$\cos\theta_{ac} = (T - a^2 - c^2 + 2\mathcal{E}_a\mathcal{E}_b)/2|\mathbf{a}||\mathbf{c}|, \qquad (A4)$$

with

$$f(S,a,b) = [S - (m_a + m_b)^2][S - (m_a - m_b)^2]. \qquad (A5)$$

In the fixed target system, the corresponding relations are

$$\omega_a = \omega(S.a,b) = (S - a^2 - b^2)/2m_b,$$

$$\omega_b = m_b, \omega_c = -\omega(U,c,b), \omega_d = -\omega(T,d,b), \qquad (A6)$$

$$|\mathbf{a}| = f^{1/2}(S,a,b)/2m_b, |\mathbf{b}| = 0,$$

$$|\mathbf{c}| = f^{1/2}(U,c,b)/2m_b, |\mathbf{d}| = f^{1/2}(T,b,d)/2m_b, \qquad (A7)$$

and

$$\cos\tilde{\theta}_{ac} = [2b^2(T - a^2 - c^2)$$

$$-(S - a^2 - b^2)(U - b^2 - c^2)]/[f(S,a,b)f(U,c,b)]^{1/2}. \qquad (A8)$$

The differential cross-sections are found from the definition of the invariant total cross-section for the interaction of two particles initially in the states $|a>$, and $|b>$ and the subsequent production of an n-particle final state, where each particle is characterized by a momentum state $|p_i>$.

This cross-section is defined as

$$\sigma(S,T) = \frac{1}{2f^{1/2}(S,a,b)(2\pi)^{3n-4}} \times \int dp_1 dp_2 ... dp_n$$

$$\prod_{i=1}^{n} \delta(p_i^2 - m_i^2)\theta(p_i) \times \delta(a + b - \sum_{i=1}^{n} p_i)\mathcal{M}(S,T,U), \qquad (A9)$$

with $\theta(p) = [(p^0/\omega) + 1]/2, \omega = (|\mathbf{p}|^2 + m^2)^{1/2}$, and

$$\mathcal{M}(S,T,U) = |<f|\mathcal{M}|a,b>|^2,$$

where $< f|M|a,b >$ is the transition amplitude from the initial to the final state. In (A9), one uses an invariant definition of the flux, which is represented as the magnitude of the relative velocity $|\mathbf{v}_a - \mathbf{v}_b|$ in the c.m. system. The flux becomes $F = f^{1/2}(s,a,b)/2\mathcal{E}_a\mathcal{E}_b$.

Particular differential cross-sections may now be obtained from (A9). Of special interest is the differential cross-section defined formally as

$$\frac{d\sigma}{dT} = \sigma(S,T)\delta[T - (a-c)^2]. \qquad (A10)$$

For scattering into the solid angle $d\Omega_{ac}$, one finds for elastic scattering in the c.m. system

$$\frac{d\sigma}{d\Omega_{c.m.}} = \frac{1}{4\pi S} f^{1/2}(S,a,b) f^{1/2}(S,c,d)\frac{d\sigma}{dT}. \qquad (A11)$$

The corresponding differential cross-section in the f.t. system may be found from (A8) and (A10) to be

$$\frac{d\sigma}{d\Omega_{f.t}} = \frac{2f^{1/2}(S,a,b)f^{3/2}(U,c,b)}{\pi g(S,T,1,m,1,m)}\frac{d\sigma}{dT}. \qquad (A12)$$

For the elastic scattering of a particle of unit mass with a particle of mass m, one finds the expression

$$g(S,t,1,m,1,m) = 128m^2[(S/4)^2 + ST/16 - (S/4)(m^2+1)/2$$

$$-(T/4)(m^2 - 1)/4 + (m^2-1)^2/16]. \qquad (A13)$$

The integration indicated in (A9) and (A10), when there is a two-particle final state characterized by four-momenta c and d, is performed in the c.m. system using the momentum-space measure

$$dcdd = |\mathbf{c}|^2 d|\mathbf{c}| d\Omega_c \frac{dc^2}{2c^0} dd \frac{dd^2}{2d^0} \qquad (A14)$$

to find (3.3).

IV. REFERENCES

[1] W. Chou, S. Dutt, T. Garavaglia, and S. K. Kauffmann, these proceedings.

[2] K. Goulianos, "Diffractive and rising cross-sections," *Comments on Nucl. Part. Phys.* **17**, No. 4, pp. 177-193 (1987).

[3] M. Honda et.al., Phys. Rev. Lett. **70**, 525 (1993).

[4] P.V. Landshoff, "Soft hadron physics," *Joint International Lepton-Photon & Europhysics Conference on High Energy Physics*, Vol. 2, pp. 365–373 (World Scientific, 1991).