Transverse Tails and Higher Order Moments*

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The tails that may be engendered in a beam's transverse phase space distribution by, *e.g.*, intrabunch wakefields and nonlinear magnetic fields, are an important diagnostic and object of tuning in linear colliders. Wire scanners or phosphorescent screen monitors yield one dimensional projected spatial profiles of such beams that are generically asymmetric around their centroids, and therefore require characterization by the third moment $\langle x^3 \rangle$ in addition to the conventional mean-square or second moment. A set of measurements spread over sufficient phase advance then allows the complete set $\langle x^3 \rangle$, $\langle x x'^2 \rangle$, $\langle x'^3 \rangle$, and $\langle x^2 x' \rangle$ to be deduced—the natural extension of the well-known 'emittance measurement' treatment of second moments. The four third moments may be usefully decomposed into parts rotating in phase space at the β-tron frequency and at its third harmonic, each specified by a phase-advance-invariant amplitude and a phase. They provide a framework for the analysis and tuning of transverse wakefield tails.

Third Moments

The totally symmetric tensor of third moments $\langle x_i x_j x_k \rangle$ ($i = 1, \ldots, d$, the number of phase space dimensions) evolves in a linear beamline [1] with transfer matrix R according to

$$\langle x_i x_j x_k \rangle \rightarrow R_{ii'} R_{jj'} R_{kk'} \langle x_{i'} x_{j'} x_{k'} \rangle \tag{1}$$

In normal coordinates for 2-dimensional phase space $\hat{x} = \frac{1}{\sqrt{\beta}} \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix} x$, and

$$\langle \hat{x}_i \hat{x}_j \hat{x}_k \rangle \rightarrow \mathcal{O}_{ii'} \mathcal{O}_{jj'} \mathcal{O}_{kk'} \langle \hat{x}_{i'} \hat{x}_{j'} \hat{x}_{k'} \rangle \tag{2}$$

where $\mathcal{O}$ is a 2×2 rotation matrix.

In two phase space dimensions $\langle x_i x_j x_k \rangle$ has 4-components, *viz.*,

$$\langle x^3 \rangle, \ \langle x x'^2 \rangle, \ \langle x'^3 \rangle, \ \langle x^2 x' \rangle \tag{3}$$

Four skew $\langle \hat{x}^3 \rangle = \langle x^3 \rangle / \beta^{3/2}$ measurements are necessary and sufficient to determine the four independent components $\langle \hat{x}_i \hat{x}_j \hat{x}_k \rangle_0$ at some reference point defined as being at phase advance $\Delta\psi = 0$

$$\begin{aligned} \langle \hat{x}^3 \rangle &= \cos^3(\Delta\psi) \langle \hat{x}^3 \rangle_0 + 3\cos(\Delta\psi)\sin^2(\Delta\psi) \langle \hat{x}\hat{x}'^2 \rangle_0 \\ &\quad + \sin^3(\Delta\psi) \langle \hat{x}'^3 \rangle_0 + 3\sin(\Delta\psi)\cos^2(\Delta\psi) \langle \hat{x}^2\hat{x}' \rangle_0 \qquad (4) \\ &= \frac{1}{4}\cos(3\Delta\psi) \left(\langle \hat{x}^3 \rangle_0 - 3 \langle \hat{x}\hat{x}'^2 \rangle_0 \right) \\ &\quad - \frac{1}{4}\sin(3\Delta\psi) \left(\langle \hat{x}'^3 \rangle_0 - 3 \langle \hat{x}^2\hat{x}' \rangle_0 \right) \\ &\quad + \frac{3}{4}\cos(\Delta\psi) \left(\langle \hat{x}^3 \rangle_0 + \langle \hat{x}\hat{x}'^2 \rangle_0 \right) \\ &\quad + \frac{3}{4}\sin(\Delta\psi) \left(\langle \hat{x}'^3 \rangle_0 + \langle \hat{x}^2\hat{x}' \rangle_0 \right) \end{aligned} \tag{5}$$

The second, Fourier series expression, shows that first and third β-tron harmonics enter.

The linear combinations

$$\langle \hat{x}^3 \rangle + \langle \hat{x}\hat{x}'^2 \rangle = \tau \sin\Psi \tag{6}$$

$$\langle \hat{x}'^3 \rangle + \langle \hat{x}^2\hat{x}' \rangle = \tau \cos\Psi \tag{7}$$

transform analogously to $\langle x_i \rangle$, *i.e.*, $\Psi \rightarrow \Psi + \Delta\psi$ along the beamline and $\tau \geq 0$ is invariant. The complementary linear combinations

$$\langle \hat{x}^3 \rangle - 3 \langle \hat{x}\hat{x}'^2 \rangle = -\kappa \sin\Phi \tag{8}$$

$$\langle \hat{x}'^3 \rangle - 3 \langle \hat{x}^2\hat{x}' \rangle = \kappa \cos\Phi \tag{9}$$

rotate at three times the normal phase advance, *i.e.*, $\Phi \rightarrow \Phi + 3\Delta\psi$ along the beamline, with $\kappa \geq 0$ invariant.

There are no symplectic invariants (like emittance) for odd-order moments, but there are useful phase-advance-invariants (*i.e.*, invariants with respect to the rotational sub-group of the full symplectic group), analogous to $\frac{1}{2}\,\mathrm{tr} \left\langle \hat{x}\hat{x}^T \right\rangle = \bar{\epsilon}$, the matched-equivalent emittance. They are in correspondence with the irreducible representations of the rotation group $SO(2)$ contained in the moments tensor.

The components can then be parameterized

$$\langle \hat{x}^3 \rangle = \frac{3}{4}\tau \sin\Psi - \frac{1}{4}\kappa \sin\Phi \tag{10}$$

$$\langle \hat{x}\hat{x}'^2 \rangle = \frac{1}{4}\tau \sin\Psi + \frac{1}{4}\kappa \sin\Phi \tag{11}$$

$$\langle \hat{x}'^3 \rangle = \frac{3}{4}\tau \cos\Psi + \frac{1}{4}\kappa \cos\Phi \tag{12}$$

$$\langle \hat{x}^2\hat{x}' \rangle = \frac{1}{4}\tau \cos\Psi - \frac{1}{4}\kappa \cos\Phi \tag{13}$$

$\langle \hat{x}'^3 \rangle$ and $\langle \hat{x}^3 \rangle$ are related by a 90° phase shift, and

$$\langle \hat{x}\hat{x}'^2 \rangle = -\frac{1}{3}\frac{d}{d\psi} \langle \hat{x}'^3 \rangle \tag{14}$$

$$\langle \hat{x}^2\hat{x}' \rangle = \frac{1}{3}\frac{d}{d\psi} \langle \hat{x}^3 \rangle \tag{15}$$

analogous to the 2nd moment relation $\langle \hat{x}\hat{x}' \rangle = \frac{1}{2} d \langle \hat{x}^2 \rangle / d\psi$.

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Sampling the observable

$$\frac{\langle x^3\rangle}{\beta^{3/2}} = \frac{3}{4}\tau\sin(\Delta\psi+\Psi) - \frac{1}{4}\kappa\sin(3\Delta\psi+\Phi) \quad (16)$$

at at least 4 appropriate phases then makes possible the determination of the two invariants and two phases that are equivalent to knowing the 4 independent moments at a point.

A 'simple' asymmetric phase space tail is characterized as having 4 third moments with the property that there is some (rigid) phase rotation that reduces 3 of them simultaneously to zero. The rotation can be chosen such that the non-vanishing moment is $\langle x'^3\rangle$, and the associated beamline phase regarded as an effective phase at which the skew or asymmetric tail originated in an impulse or kick. In invariant terms a 'simple' tail has $\kappa = \tau$ and some phase where $\Phi = \Psi = 0$, *i.e.*, $\Phi = 3\Psi$ (*modulo* 2π).

More generally it is useful to identify a 'principal axis' phase P at which $\langle \hat{x}'^3\rangle$ is maximized, and hence $\langle \hat{x}\hat{x}'^2\rangle_P = 0$; the magnitude of the $\langle \hat{x}^3\rangle_P$ and $\langle \hat{x}^2\hat{x}'\rangle_P < \langle \hat{x}'^3\rangle_P$ moments then measures how 'simple' the tail is. For a 'simple' tail $\langle \hat{x}'^3\rangle_P = \kappa = \tau$.

In 4-dimensional phase space there are 12 additional third moments ($\langle x^2y\rangle$, $\langle xx'y\rangle$, *etc.*), that are in principle accessible through the observable

$$\langle x^2y\rangle + \langle xy^2\rangle = \frac{1}{3}\left[\langle u^3\rangle - \frac{\sqrt{2}}{4}\left(\langle x^3\rangle + \langle y^3\rangle\right)\right] \quad (17)$$

containing Fourier components for $\Delta\psi_x$, $\Delta\psi_y$, $2\Delta\psi_x \pm \Delta\psi_y$, and $\Delta\psi_x \pm 2\Delta\psi_y$, and corresponding to irreducible representations of $SO(2)\times SO(2)$. Thus observations must be made at at least 12 phase-advance-incommensurate beamline locations for a complete 4-dimensional analysis to be made. There are 5 4th moments in 2 phase space dimensions, and 35 in 4 dimensions.

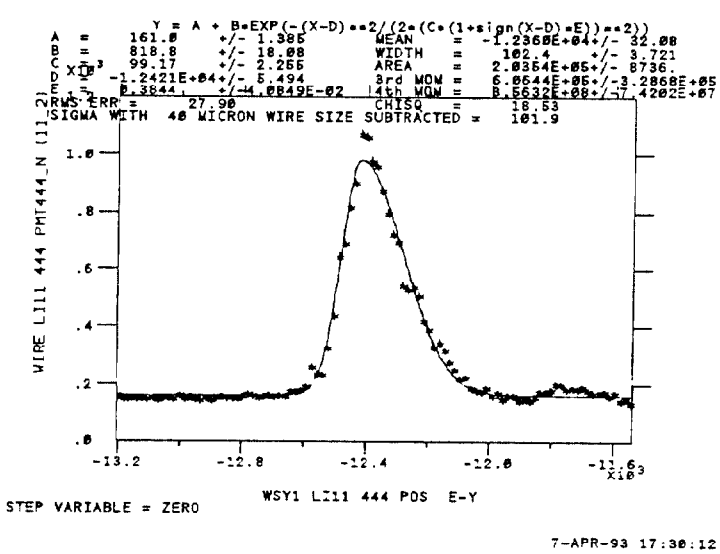

FIG. 1 An 'asymmetric Gaussian' fit to SLC wire scanner data, indicating a wakefield tail.

Asymmetric Gaussian-like Function

A simple and useful beam profile fitting function can be constructed by allowing the width of a Gaussian function to be different on the right from on the left

$$\sigma_L = \sigma_0(1-E),\ \sigma_R = \sigma_0(1+E) \quad (18)$$

$(\sigma_L + \sigma_R)/2 = \sigma_0$ as for a (symmetric) Gaussian, and the asymmetry $(\sigma_R - \sigma_L)/(\sigma_R + \sigma_L) = E$ is bounded: $-1 \le E \le 1$. Fitting (Fig. 1) to the five parameter function (including a background offset)

$$f(x) = A + B\exp\left\{-\frac{1}{2}\left[\frac{x-D}{C\left(1+\operatorname{sgn}(x-D)\,E\right)}\right]^2\right\} \quad (19)$$

yields $\left\langle (x-\langle x\rangle)^3\right\rangle = 2\sqrt{\frac{2}{\pi}}\left[1+\left(\frac{16}{\pi}-5\right)E^2\right]EC^3$.

Phenomenology of Transverse Wakefield Tails

In the case of wakefields the asymmetry arises from the fact that kicks are longitudinally differential, *i.e.*, $\Delta x'_\tau = (e^2/E)\int dN_{\tau_0}\, W_\perp(\tau-\tau_0)\,(x_{\tau_0}-\xi)$, where the impulsive point dipole wakefield $W_\perp(\Delta\tau)$ depends on the longitudinal coordinate difference $\Delta\tau$, N is the bunch population, E the beam energy, and ξ the structure offset. A 'simple' tail corresponds to a single impulsive excitation or series of multiple excitations that are sufficiently weak that, being dominantly first order or 'two-particle' like, they obey vector addition in phase space (Fig. 2a).

The exact relationship between third moments or tails and rms emittance depends on the details of the mechanism—second and third moments are *a priori* independent. In the simple though qualitatively representative model in which the wakefield coefficient depends linearly on longitudinal distance $W_\perp = W'_\perp\theta(\tau-\tau_0)\,(\tau-\tau_0)$, the matched-equivalent emittance $\bar{\epsilon}$ [2] of a 'simple' wakefield tail

$$\bar{\epsilon} \cong \begin{pmatrix} 0.29069 \\ 4/15 \end{pmatrix} \frac{\beta}{2}\left(\frac{e^2N}{E}W'_\perp\xi\right)^2 \langle\tau^2\rangle \quad (20)$$

$$\cong \begin{pmatrix} 0.39010 \\ (70/67)^{2/3}\cdot 4/5 \end{pmatrix} \left|\langle \hat{x}'^3\rangle_P\right|^{2/3} \quad (21)$$

The upper geometric factor corresponds to a Gaussian and the lower to a uniform (step) longitudinal distribution, and $\langle\tau^2\rangle$ is the mean square bunch length.

For a prior emittance ϵ_0, the net prompt emittance $\tilde{\epsilon} = \sqrt{\epsilon_0^2 + 2\epsilon_0\bar{\epsilon}}$, growing still further to $\bar{\tilde{\epsilon}} = \epsilon_0 + \bar{\epsilon}$ if filamentation subsequently occurs. The significance of relating the emittance due to wakefields to the *third* moments is that it connects observable quantities—the third moment, unlike the second, can be expected to be due entirely to the transverse wakefield.

The Lorentz invariants $\gamma^{3/2}\langle \hat{x}_i\hat{x}_j\hat{x}_k\rangle$, and $\gamma^{3/2}\tau$, $\gamma^{3/2}\kappa$ are usually most convenient to deal with. It is useful to note that for $\gamma^{3/2}\langle \hat{x}^3\rangle_P = 10^{-7}\,\mathrm{m}^{3/2} = 10^{11}\,(\mu\text{m-}\mu\text{rad})^{3/2}$, the maximum fractional skew $\left|\langle x^3\rangle_{\max}/\langle x^2\rangle^{3/2}\right| = \gamma^{3/2}\langle \hat{x}'^3\rangle_P/(\gamma\bar{\epsilon})^{3/2} = 1$ for an invariant emittance $\gamma\bar{\epsilon} \cong 2.15\cdot 10^{-5}\,\mathrm{m}$, and that $\gamma\bar{\epsilon} \cong 0.84\cdot 10^{-5}\,\mathrm{m}$ for a 'simple'

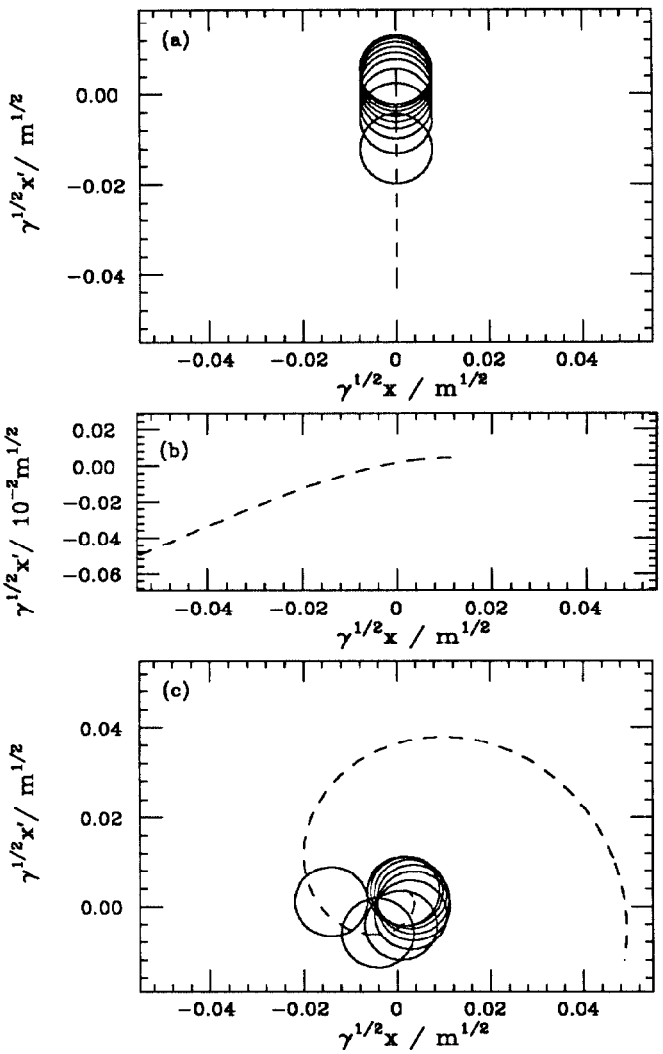

FIG. 2 Phase space distributions in beam centroid subtracted normal coordinates in which the transverse centroids for definite longitudinal 'slices' are indicated by dashed lines, and longitudinal segments, each containing 1/9 of the total beam charge, are shown as circles corresponding to a transverse emittance $\gamma\epsilon_0 = 3 \cdot 10^{-5}$ m.

(a) $\gamma^{3/2}\kappa = \gamma^{3/2}\tau = 2.3 \cdot 10^{-7}\,\mathrm{m}^{3/2}$, $\gamma\bar{\epsilon} = 1.4 \cdot 10^{-5}$ m, and $\gamma\epsilon = 0$ [3].

(b) $\gamma^{3/2}\kappa \approx \gamma^{3/2}\tau = 18 \cdot 10^{-7}\,\mathrm{m}^{3/2}$, $\gamma\bar{\epsilon} = 5.8 \cdot 10^{-5}$ m, and $\gamma\epsilon = 0.02 \cdot 10^{-5}$ m.

(c) $\gamma^{3/2}\tau = 4.8 \cdot 10^{-7}\,\mathrm{m}^{3/2}$, $\gamma^{3/2}\kappa = 0.9 \cdot 10^{-7}\,\mathrm{m}^{3/2}$, $\gamma\bar{\epsilon} = 2.4 \cdot 10^{-5}$ m, and $\gamma\epsilon = 2.3 \cdot 10^{-5}$ m.

tail. Fractional skew and emittance are in general related by

$$\begin{pmatrix} 0.39010 \\ (70/67)^{2/3} \cdot 4/5 \end{pmatrix} \left[\frac{\gamma^{3/2} \langle \hat{x}'^3 \rangle_P}{(\gamma\tilde{\epsilon})^{3/2}} \right]^{2/3} = \frac{1}{2} \left(\frac{\tilde{\epsilon}}{\epsilon_0} - \frac{\epsilon_0}{\tilde{\epsilon}} \right) \rightarrow \begin{cases} (\tilde{\epsilon} - \epsilon_0)/\epsilon_0, & \tilde{\epsilon} - \epsilon_0 \ll \epsilon_0 \\ \frac{1}{2}\tilde{\epsilon}/\epsilon_0, & \tilde{\epsilon} \gg \epsilon_0 \end{cases} \tag{22}$$

If the transverse offsets relative to the bunch centroid of significant fractions of the charge become comparable to the offset of the centroid relative to the center of cylindrical symmetry of the structure, the tail will deviate from the 'simple' form (Fig. 2b). Nevertheless for a (linearly) increasing point wakefield function, the tensor of third moments will appear 'simple' in as much as the third moments will be dominated by particles that get deflected to large phase space distances—far from the bulk of the charge—and are thus insensitive to its precise distribution relative to the center of the structure. However, the cumulative transverse phase advance differential that will arise in the presence of magnetic lattice chromaticity if the bunch has an energy spread correlated with longitudinal position (as in BNS damping [4]), creates significant departures from the 'simple' form (Fig. 2c). The third moments will be Landau damped, although when taken about the beam centroid they may manifest transient increases as the first moment vector (*i.e.*, the centroid) damps. The Fourier component $\tau > \kappa$ (becoming $\tau \gg \kappa$) is characteristic of this process, since the κ component oscillates and hence damps at three times the rate for τ (*i.e.*, three times the β-tron frequency).

At the SLC we have recently implemented on-line analysis of third moments using the fit (19) to wire scanner profiles, and eqn. (16). The use of linac orbit bumps to cancel wakefield excitation [5], [6] can be enhanced both in speed and efficacy by knowing the 'tail' phase and amplitude changes resulting from test variations in the orbit. Knowledge of the tail nature ('simple' or not?) is important in that bumps that fail to drive the tail closer toward 'simplicity' are too distant (downstream) from the source to remove it fully. First efforts at automating the tuning procedure using digital feedback are in progress.

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1. There are no wakefield or chromaticity effects in the beamline considered here, as is usually true to a good approximation in a high energy linac if it is sufficiently short.
2. $\bar{\epsilon} = \frac{1}{2}\left[\left\langle \langle \hat{x} \rangle_\perp^2 \right\rangle_\tau + \left\langle \langle \hat{x}' \rangle_\perp^2 \right\rangle_\tau\right]$ here. $\langle\rangle_\perp$ represents a transverse phase space average, *i.e.*, $\langle x \rangle_\perp$ and $\langle x' \rangle_\perp$ are the longitudinal 'slice' centroids, and $\langle\rangle_\tau$ the longitudinal beam average.
3. The emittance due to wakefields, or emittance if the emittance in the absence of wakefields is zero, $\epsilon = \left[\left\langle \langle x \rangle_\perp^2 \right\rangle_\tau \left\langle \langle x' \rangle_\perp^2 \right\rangle_\tau - \langle \langle x \rangle_\perp \langle x' \rangle_\perp \rangle_\tau^2\right]^{1/2} \leq \bar{\epsilon}$. In general the net prompt emittance $\tilde{\epsilon} = \sqrt{\epsilon_0^2 + 2\epsilon_0\bar{\epsilon} + \epsilon^2}$, and $\epsilon \rightarrow \bar{\epsilon}$ under the influence of filamentation.

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