Longitudinal Diffusion as Inflicted by Arbitrary Band-Width Random-Modulated Currents in Feeders of Detuned Cavities

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Abstract

Diffusion coefficient for a bunched p-beam in a synchrotron is presented with two technically imposed items included into the scheme. (1) The role of basic external noises is attributed to random envelopes $i^{(a,\varphi)}(t)$ carried by two modulated time-quadrature RF-currents. These represent amplitude, (a) or (small) phase, (φ) noises of forward current in cavity feeders. (Commonly, the (a,φ)-noise voltages $v^{(a,\varphi)}(t)$ at the accelerating gaps are taken as the basic ones.) Phase shifts between spectral components $v(\omega)$ vs. $i(\omega)$ are inevitable due to both, the transfer function from feeder current to the detuned-cavity gap voltage, and the phasor diagram of the gap voltages maintained under beam loading. None of the 'pure' noises $v^{(a)}(t)$ or $v^{(\varphi)}(t)$ is observable practically. Only their mixture is. (2) The possibility of an arbitrary ratio of the noise correlation time to the revolution period (i.e., the effect of noise spectrum to stretch over many revolution frequency harmonics) is incorporated, which is important to treat the noise-induced diffusion in the large rings (UNK, LHC, SSC).

I. INTRODUCTION

A. Diffusion Equation

Longitudinal dilution of a proton bunch subjected to noise obeys a diffusion equation which according to, say, ref.[1] reads

$$\frac{\partial\langle F_0\rangle(\mathcal{J},t)}{\partial t}=\frac{\partial}{\partial\mathcal{J}}\left(D(\mathcal{J})\frac{\partial\langle F_0\rangle(\mathcal{J},t)}{\partial\mathcal{J}}\right). \quad (1)$$

Here t is time; $\mathcal{J}$ is action; F is bunch distribution; $\langle\ldots\rangle$ is statistical average over noise ensemble; subscript '0' denotes the mathematical average over phase ψ, the canonical conjugate of $\mathcal{J}$. Variables $(\psi,\mathcal{J})$ are introduced in the phase-plane $(\vartheta,\vartheta'\equiv d\vartheta/dt)$, where $\vartheta=\Theta-\omega_s t$ is azimuth in a co-rotating frame; Θ is azimuth around the ring in the laboratory frame; ω_s is the angular velocity of a reference particle synchronous to the nominal RF. The origin $\vartheta=0$ is put on the unperturbed reference particle of the bunch in question.

The diffusion coefficient is

$$D(\mathcal{J}) = A\sum_{m=-\infty}^{\infty}(mq)^2\sum_{k,k_1=-\infty}^{\infty}\frac{I^*_{mk}(\mathcal{J})}{k}\frac{I_{mk_1}(\mathcal{J})}{k_1}\times \quad (2)$$
$$\times\int_{-\infty}^{\infty}\overline{\langle\Delta V_k(t)\Delta V^*_{k_1}(t-\tau)\rangle}\exp\left(im\Omega_s(\mathcal{J})\tau\right)d\tau.$$

The beam is subjected to a random voltage $\Delta V(\vartheta,t)$ which is decomposed into $\sum_k\Delta V_k(t)e^{ik\vartheta}$, and eq.2 embeds time correlations of random amplitudes $\Delta V_k(t)$. Functions $I^*_{mk}(\mathcal{J})$ are the coefficients of series which expand a plane wave $e^{ik\vartheta(\mathcal{J},\psi)}$ into sum over multipoles: $\sum_m I^*_{mk}(\mathcal{J})e^{im\psi}$. Factor A is equal to

$$A=\frac{1}{2}\left(\frac{\Omega_0^2}{q^2V_{\rm ext}\sin\varphi_s}\right)^2. \quad (3)$$

Here Ω_0 is the small-amplitude synchrotron frequency (circular); q is the acceleration frequency harmonic number; $V_{\rm ext}$ is the nominal amplitude of accelerating voltage; φ_s is the stable phase angle ($\varphi_s>0$ below transition, the synchronous energy gain being $eV_{\rm ext}\cos\varphi_s$); $\Omega_s(\mathcal{J})=d\psi/dt$ is the non-linear synchrotron frequency.

B. Representative Time Scales

The bunch had been matched and stationary until $t=0$ when the noise was switched on. Diffusion approximation implies fluctuations $\Delta V(\vartheta,t)$ to be fast and weak:

$$\tau_{\Delta V}\ \ll\ \tau_{dif}, \quad (4)$$

where τ_{dif} is a rate measure of the noise-induced bunch dilution; $\tau_{\Delta V}$ is a correlation time of $\Delta V(\vartheta,t)$. Bunch evolution is pursued at $t\gg\tau_{\Delta V}$.

In general, $\Delta V_k(t)$ is a periodically unstationary stochastic process: its moments $\langle\Delta V_k(t)\Delta V^*_{k_1}(t-\tau)\rangle$ are periodic functions of t, $2\pi/\omega_s$ being a period. The slow diffusion is governed by the non-oscillating terms in $\langle\Delta V_k(t)\Delta V^*_{k_1}(t-\tau)\rangle$, these being extracted by t-averaging the latter over a turn (over-line in eq.2). The correlations thus smoothed depend on τ solely, and can hence be treated in terms of spectral intensities. Function $\langle F_0\rangle(\mathcal{J},t)$ is as well smoothed over a turn, being a slowly varying one:

$$2\pi/\omega_s\ \ll\ \tau_{dif}. \quad (5)$$

Refs.[2, 3, 4] study stationary noise $\Delta V(\vartheta,t)$ as given in the co-moving frame directly, which implicitly conjectures the noise's low-frequency and narrow-band features:

$$2\pi/\omega_s\ \ll\ \tau_{\Delta V}\ \ll\ \tau_{dif}. \quad (6)$$

However, with the orbit perimeters L growing (UNK, LHC, SSC) feasible are the relationships

$$2\pi/\omega_s\ \sim\ \tau_{\Delta V}\ \ll\ \tau_{dif};\quad \tau_{\Delta V}\ \ll\ 2\pi/\omega_s\ \ll\ \tau_{dif}. \quad (7)$$

This paper presents formulae which hold true not only within the range given by eq.6, but well beyond it, eqs.7.

II. NOISES OF VOLTAGE

A. Accelerating Voltage

This field is imposed by N accelerating cavities (gaps),

$$E(\Theta,t) = L^{-1}\sum\nolimits_{n=1}^{N} G_n(\Theta) V_n \cos(q'\omega_s t - \varphi_n). \tag{8}$$

Here V_n, φ_n are the amplitude and phase of voltage across the n-th gap; $q'\omega_s$ is the nominal RF — a (higher) harmonic of the acceleration frequency $q\omega_s$, $q'/q = 1, 2, \ldots$

Functions $G_n(\Theta)$ specify the field localization. They can be decomposed into Fourier series $\sum_k G_{n;k} e^{ik\Theta}$. Given $\int_0^{2\pi} |G_n(\Theta)| d\Theta = 2\pi$, quantities $|G_{n;k}|$ grow into transit-time factors at $\omega = k\omega_s$. Variables V_n, φ_n are adjusted so as to provide propagating wave $V'_{\rm ext}\cos(q'\vartheta + \varphi'_s)$, stationary w.r.t. the co-rotating frame. The main RF system drives a wave whose $(V'_{\rm ext}, q', \varphi'_s) = (V_{\rm ext}, q, \varphi_s)$.

B. Random Voltages

Put down the random field by analogy with eq.8 as

$$\Delta E(\Theta,t) = L^{-1}\sum\nolimits_{n=1}^{N} G_n(\Theta)\, u_n(t) \tag{9}$$

with $u_n(t)$ being the noise voltage. Take the latter as

$$u_n(t) = \sum\nolimits_{\zeta} v_n^{(\zeta)}(t) \cos(q'\omega_s t - \varphi_n^{(\zeta)}). \tag{10}$$

Here ζ is a noise type index. Carrier phase $\varphi_n^{(\zeta)}$ is, generally, other than φ_n from eq.8. Modulating voltages $v_n^{(\zeta)}(t)$, $\langle v_n^{(\zeta)}(t)\rangle = 0$ are the stochastic processes, mutually stationary w.r.t. the laboratory frame.

The particular option of indices $\zeta = a, \varphi$ and phases

$$\varphi_n^{(a)} = \varphi_n; \quad \varphi_n^{(\varphi)} = \varphi_n - \pi/2 \tag{11}$$

allows one to interpret eq.10 as a decomposition of $u_n(t)$ into a sum of its inphase ($\zeta = a$) and quadrature ($\zeta = \varphi$) components w.r.t. the reference signal, eq.8. The inphase component represents the noises of amplitude, while the quadrature one — those of phase,

$$v_n^{(a)}(t) = \Delta V_n(t); \quad v_n^{(\varphi)}(t) = V_n \Delta\varphi_n(t), \tag{12}$$

of the gap voltages. Either may be driven in a relatively independent way (noises in a master oscillator, an amplitude modulator, a phase shifter).

Adopting $q' = \varphi_n^{(\zeta)} = 0$ and, e.g., $\zeta = 1$ results in a particular case of gap noise voltage $u_n(t) = v_n^{(\zeta)}(t)$, stationary w.r.t. the laboratory frame (a shot noise of anode DC current in the tube, a ripple of its power supply).

C. Diffusion as Inflicted by Random Voltages

Inserting eq.10 into eq.2 yields

$$D(\mathcal{J}) = A \sum\nolimits_{n,n_1=1}^{N} \sum\nolimits_{\zeta,\zeta_1} \sum\nolimits_{k,m=-\infty}^{\infty} \times \tag{13}$$
$$\times\, P_{nn_1}^{(\zeta\zeta_1)}(k\omega_s + m\Omega_s(\mathcal{J}))\; \mathcal{V}_{n;mk}^{(\zeta)}(\mathcal{J})\; \mathcal{V}_{n_1;mk}^{(\zeta_1)*}(\mathcal{J}).$$

The (mutual) spectral power densities $P_{nn_1}^{(\zeta\zeta_1)}(\omega)$ are the Fourier transforms of voltage cross-correlations,

$$P_{nn_1}^{(\zeta\zeta_1)}(\omega) = \int_{-\infty}^{\infty} \langle v_n^{(\zeta)}(t)\, v_{n_1}^{(\zeta_1)}(t-\tau)\rangle\, e^{i\omega\tau}\, d\tau. \tag{14}$$

Weight factors $\mathcal{V}_{n;mk}^{(\zeta)}(\mathcal{J})$ specify the bunch excitation at the m-th multipole caused by spectral components of noise $v_n^{(\zeta)}(t)$ at frequency $\omega \simeq k\omega_s$:

$$\begin{aligned} \mathcal{V}_{n;mk}^{(\zeta)}(\mathcal{J}) &= (mq/2) \times \\ &\times \left(\frac{I^*_{m,k+q'}(\mathcal{J})}{k+q'}\, G_{n;k+q'}\, e^{+i\varphi_n^{(\zeta)}} + \right. \\ &\left. + \frac{I^*_{m,k-q'}(\mathcal{J})}{k-q'}\, G_{n;k-q'}\, e^{-i\varphi_n^{(\zeta)}} \right). \end{aligned} \tag{15}$$

These functions depend on the carrier frequency $q'\omega_s$ and its phase $\varphi_n^{(\zeta)}$ w.r.t. the bunched beam. The presence of expansion coefficients $I^*_{m,k\pm q'}$ accompanied by transit-time factors $G_{n;k\pm q'}$ is quite explainable. Multiplication of $v_n^{(\zeta)}(t)$ by a high-frequency oscillation $\cos(q'\omega_s t - \varphi_n^{(\zeta)})$ translates spectral components $v_n^{(\zeta)}(t)$ from $\omega \simeq k\omega_s$ into a region of (higher) frequencies $\omega \simeq (k \pm q')\omega_s$ from which these affect the beam by driving its multipole oscillations.

For example, consider one gap ($n = N = 1$), a single noise source $v_n^{(\zeta)}(t)$ with $\zeta = a$ or φ whose spectrum is localized in a lower-frequency domain within a bandwidth $\Delta\omega_v \ll \omega_s$ so as to comply with eq.6:

$$P_{nn_1}^{(\zeta\zeta_1)}(k\omega_s + \Omega) \simeq P_n^{(\zeta)}(\Omega)\delta_{nn_1}\delta_{\zeta\zeta_1}\delta_{k0} \tag{16}$$

with δ_{ij} being the Kronecker's delta-symbol. In this case eqs.13, 15 yield the results of refs.[2, 3, 4].

III. NOISES OF CURRENT

Noises of accelerating voltage are excited by those of RF-drive current in the gap feeders. The latter noises are more logical to deal with while specifying the noise tolerances on a practical, RF-engineering level.

A. Drive Current and Beam Loading Effect

To excite accelerating field, eq.8, the RF-generator drives a forward wave of current in the gap feeder. Let this wave be presented by a current

$$I_n \cos(q'\omega_s t - \phi_n) \tag{17}$$

which flows through, say, a coupling device between the feeder and the n-th gap. (Reflected-wave current does not enter this definition.)

The following phasor diagram of voltages is maintained under beam loading of the narrow-band gap

$$V_n e^{i\varphi_n} = T_n(q'\omega_s)\, I_n e^{i\phi_n} - 2T'_n(q'\omega_s)\, G_{n;-q'} J_{q'}. \tag{18}$$

Here $T_n(\omega)$ is a transfer function between Fourier transforms of the feeder current and the gap voltage thereby excited. Transfer function $T'_n(\omega)$ from the beam current to beam-induced voltage is such as to have $\mathrm{Re}T'_n(\omega) > 0$, and $T'_n(\omega)|G_{n;q'}|^2$ to become a standard beam-gap coupling impedance at $\omega \simeq q'\omega_s$. Coefficient $J_{q'}$ is the one from Fourier series $J(\vartheta) = \sum_k J_k e^{ik\vartheta}$ with $J(\vartheta)$ being the stationary beam current. Given short bunches, $J_{q'} \simeq J_0$, where J_0 is the average beam current. Phase ϕ_n required to calculate $D(\mathcal{J})$ is readily extracted from eq.18:

$$\phi_n = \arg \frac{V_n e^{i\varphi_n} + 2T'_n(q'\omega_s)\, G_{n;-q'} J_{q'}}{T_n(q'\omega_s)\, I_n}, \tag{19}$$

and $\phi_n \to \varphi_n - \arg T_n(q'\omega_s)$ when $J_{q'} \to 0$.

Standing-wave accelerating cavities are the most widely used ones, in which case

$$T_n(\omega) = \frac{2}{g} T'_n(\omega);\ T'_n(\omega) = R_n \left(1 - i\frac{\omega^2 - \omega_{0n}^2}{2\omega\Delta\omega_n}\right)^{-1}, \tag{20}$$

where R_n, $\Delta\omega_n$ are the shunt impedance and bandwidth (both loaded ones); ω_{0n} is the resonant frequency; g is the gain in coupling-loop to accelerating-gap voltage transformation. An optimal cavity detuning ($\omega_{0n} \neq q'\omega_s$) exists which would offer a resistive load to the RF-generator and, hence, result in $\phi_n = \varphi_n$.

B. Random Currents

Noises in RF-feeding circuitry induce random addition $j_n(t)$ to the main current, eq.17. It drives voltage $u_n(t)$ across the n-th gap, eq.9. A feeder, a coupler and a gap are a linear and stationary circuit. Hence, the relation between $j_n(t)$ and $u_n(t)$ is linear and time-invariant:

$$u_n(\omega) = T_n(\omega)\, j_n(\omega) \tag{21}$$

with $T_n(\omega)$ being the same as in eqs.18, 20. To comply with eq.10, consider random-modulated current

$$j_n(t) = \sum\nolimits_\zeta i_n^{(\zeta)}(t) \cos(q'\omega_s t - \phi_n^{(\zeta)}) \tag{22}$$

with $i_n^{(\zeta)}(t)$, $\langle i_n^{(\zeta)}(t)\rangle = 0$ being the stochastic processes, mutually stationary w.r.t. the laboratory frame.

Technically imposed system of basic perturbations is that of the amplitude and phase noises of drive current, eq.17, their cross-uncorrelated performance being quite feasible practically. To get these noises take $\zeta = a', \varphi'$ along with

$$\phi_n^{(a')} = \phi_n;\quad \phi_n^{(\varphi')} = \phi_n - \pi/2, \tag{23}$$
$$i_n^{(a')}(t) = \Delta I_n(t);\quad i_n^{(\varphi')}(t) = I_n \Delta\phi_n(t). \tag{24}$$

By taking $q' = \phi_n^{(\zeta)} = 0$ and, e.g., $\zeta = 1$ one arrives at the noise current $j_n(t) = i_n^{(\zeta)}(t)$, stationary w.r.t. the laboratory frame.

C. Diffusion as Inflicted by Random Currents

Using eq.21 one finds relation between the Fourier transforms of $v, i_n^{(\zeta)}(t)$:

$$v_n^{(\zeta)}(\omega)\, e^{\pm i\varphi_n^{(\zeta)}} = T_n(\omega \pm q'\omega_s)\, i_n^{(\zeta)}(\omega)\, e^{\pm i\phi_n^{(\zeta)}}. \tag{25}$$

A 'weak' stationarity of $v, i_n^{(\zeta)}(t)$ implies:

$$\langle v_n^{(\zeta)}(\omega)\, v_{n_1}^{(\zeta_1)*}(\omega_1)\rangle = 2\pi\, P_{nn_1}^{(\zeta\zeta_1)}(\omega)\, \delta(\omega - \omega_1); \tag{26}$$
$$\langle i_n^{(\zeta)}(\omega)\, i_{n_1}^{(\zeta_1)*}(\omega_1)\rangle = 2\pi\, Q_{nn_1}^{(\zeta\zeta_1)}(\omega)\, \delta(\omega - \omega_1). \tag{27}$$

Here $P_{nn_1}^{(\zeta\zeta_1)}(\omega)$ is the voltage noise spectrum, eq.14, while $Q_{nn_1}^{(\zeta\zeta_1)}(\omega)$ is that for the current noise; $\delta(\omega)$ is the delta-function. Eqs.25, 26, 27 allow one to express $P_{nn_1}^{(\zeta\zeta_1)}(\omega)$ through $Q_{nn_1}^{(\zeta\zeta_1)}(\omega)$. Inserting them into eqs.13, 15 yields

$$D(\mathcal{J}) = A \sum\nolimits_{n,n_1=1}^{N} \sum\nolimits_{\zeta,\zeta_1} \sum\nolimits_{k,m=-\infty}^{\infty} \times \tag{28}$$
$$\times\ Q_{nn_1}^{(\zeta\zeta_1)}(k\omega_s + m\Omega_s(\mathcal{J}))\, \mathcal{U}_{n;mk}^{(\zeta)}(\mathcal{J})\, \mathcal{U}_{n_1;mk}^{(\zeta_1)*}(\mathcal{J})$$

with 'current-wise' weight factors, cf. eq.15,

$$\mathcal{U}_{n;mk}^{(\zeta)}(\mathcal{J}) = (mq/2) \times \tag{29}$$
$$\times \left(\frac{I^*_{m,k+q'}(\mathcal{J})}{k+q'}\, G_{n;k+q'}\, T_n\left[(k+q')\omega_s + m\Omega_s(\mathcal{J})\right] e^{+i\phi_n^{(\zeta)}} + \right.$$
$$\left. + \frac{I^*_{m,k-q'}(\mathcal{J})}{k-q'}\, G_{n;k-q'}\, T_n\left[(k-q')\omega_s + m\Omega_s(\mathcal{J})\right] e^{-i\phi_n^{(\zeta)}} \right).$$

Commonly, in beam dynamics studies, refs.[2, 3, 4], amplitude and phase noises of the gap voltages are treated as the basic ones, in which case eq.11 and $P_{nn_1}^{(a\varphi)} = P_{nn_1}^{(\varphi a)} = 0$ should be inserted into eqs.13, 15. However, from a practical standpoint the use of eqs.28, 29 with eq.23 and $Q_{nn_1}^{(a'\varphi')} = Q_{nn_1}^{(\varphi' a')} = 0$ looks more adequate. Thus, quite different expressions for $D(\mathcal{J})$ are arrived at.

For stationary noises $u, j_n(t)$, whose $q' = 0$, $\varphi_n^{(\zeta)} = 0$, $\phi_n^{(\zeta)} = 0$, and $\zeta = 1$, the entire distinction between eqs.13, 15 and eqs.28, 29 is reducible to a mere noise filtering: $P_{nn_1}^{(\zeta\zeta_1)}(\omega) = T_n(\omega) T^*_{n_1}(\omega) Q_{nn_1}^{(\zeta\zeta_1)}(\omega)$.

The author thanks Drs. V. Balbekov and G. Gurov for the instructive discussions on the subject.

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