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Analytic Criteria for Stability of Beam Loaded R.F. Systems

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Abstract

This paper presents the instability analysis of a beamloaded radio-frequency system with beam phase-loop and cavity tuning-loop for both accelerating and nonaccelerating beams. The case of voltage-proportional feedback around the cavity is also included. The symbolic manipulation program SMP [1] was used to expand and simplify the Routh determinantal conditions for a fifth order characteristic polynomial. The paper is a much abridged version of an internal design note [2].





The disposition of steady state phasors is as shown above. We adopt the notation of Reference [2]. The cavity voltage is $\mathbf{V}(t)e^{j\omega t}$ and the total current driving the cavity is $\mathbf{I}_T(t)e^{j\omega t}$, where t indicates time and ω is the drive angular frequency. Bold face indicates complex quantities, and ordinary type denotes scalars. We employ dot notation for time derivatives. The cavity fundamental resonance is modelled as a parallel resonance LCR circuit. Let $\Omega_{res} = 1/\sqrt{LC}$ be the resonance frequency and $\alpha =$ $\Omega_{res}/(2Q) = 1/(2RC)$ be the half-bandwidth. We write the voltage and current as the sum of steady state parts $\mathbf{V}^0 = V^0 e^{j\psi_V}$ and $\mathbf{I}_T^0 = I_T^0 e^{j\psi_T}$, and small time dependent perturbations. We use ψ to denote steady state phases and ϕ perturbation phases. Let $\Psi = \psi_V - \psi_T$.

A. Steady state

We must specify the steady state generator current $\mathbf{I}_g^0 = I_g^0 e^{j\psi_g}$ and beam image current $\mathbf{I}_b^0 = I_b^0 e^{j\psi_b}$ which sum to form the total current \mathbf{I}_T^0 . The beam current is $\approx 90^\circ$ out of phase with the cavity voltage; depending on the synchronous phase angle μ_b . We set $\mu_b = 0$ for a non-accelerating beam. Hence $\psi_b = \pm (\pi/2 + \mu_b)$ and the – sign applies below transition energy and the + above. We adopt the dimensionless current ratios $Y_g = I_g^0/I_V^0$ and $Y_b = I_b^0/I_V^0$, where $I_V^0 = V^0/R$. The components obey:

$$1 = Y_g \cos \psi_g - Y_b \sin \mu_b$$

$$\tan \Psi = Y_b \cos \mu_b - Y_g \sin \psi_g .$$

From this follows the detuning $\tan \Psi = (\Omega^2 - \omega^2)/2\alpha\omega$. Until we choose a definite value for ψ_g , there is no direct relation between $\tan \Psi$ and (Y_b, μ_b) .

B. Non steady state

Let us assume the "slow approximation" $\ddot{\mathbf{V}} \ll \omega \dot{\mathbf{V}}$ and $\dot{\mathbf{I}}_T \ll \omega \mathbf{I}_T$. We allow for a varying resonance frequency $\Omega(t) = \Omega_0(t) + \Delta \Omega(t)$. We introduce the perturbation vectors \mathbf{e} as follows: $\mathbf{V} = \mathbf{V}^0(1 + \mathbf{e}_V)$ and $\mathbf{I}_T^0(1 + \mathbf{e}_T) =$ $= \mathbf{I}_g^0(1 + \mathbf{e}_g) + \mathbf{I}_b^0(1 + \mathbf{e}_b)$. The dimensionless components z_r and ϕ_r of the vector $\mathbf{e}_r = (z_r + j\phi_r)$ are amplitude and phase modulations, respectively. The cavity response is modelled by:

$$\begin{aligned} z_V(1+s\tau_c) + \phi_V \tan \Psi + Y_g(\phi_g \sin \psi_g - z_g \cos \psi_g) + \\ + Y_b(z_b \sin \mu_b - \phi_b \cos \mu_b) &= 0 , \\ \phi_V(1+s\tau_c) - z_V \tan \Psi - Y_g(\phi_g \cos \psi_g + z_g \sin \psi_g) + \\ + Y_b(z_b \cos \mu_b + \phi_b \sin \mu_b) &= \tau_c \Delta \Omega . \end{aligned}$$

Here $\tau_c = \alpha^{-1}$ is the cavity time constant, and time derivatives are replaced by the Laplace operator s.

C. Beam rigid bunch dipole motion

Suppose the ideal drive frequency is synchronous with a particle travelling with the equilibrium. However, as a result of modulations the cavity phase may advance or lag by an amount ϕ_V . Likewise, the beam centroid may differ from the ideal phase by an amount ϕ_b . Suppose the cavity has relative amplitude modulation z_V .

To first order in perturbation amplitudes, the Laplace transform, of the beam energy deviation δE is:

$$s\,\delta E = K_1[z_V\sin\mu_b + (\phi_V - \phi_b)\cos\mu_b] \;.$$

Because of the energy deviation, the phase error ϕ_b will advance at the rate: $s \phi_b = K_2 \delta E$. The product $\sqrt{K_1 \times K_2} = \Omega_s$, the synchrotron frequency sans the usual $\cos \mu_b$ term.

D. Stability conditions

The system response contains only self-damped oscillations, when all zeros of the characteristic polynomial lie in the left half of the complex plane. Necessary conditions are for the coefficients of s^n and the Routh-Hurwitz criteria [RH(i) for i = 1, 2, ..., n+1] for combinations of the coefficients to be greater than zero. We shall omit trivial conditions such as $\tau_c > 0$.

II. CAVITY AND BEAM DIPOLE MODE

This is the case originally treated by Robinson [3]. The model assumes that the generator current is maintained by an ideal feed-forward.

Characteristic polynomial

$$\begin{split} \Omega_s^2 [\cos \mu_b \sec^2 \Psi - Y_b \tan \Psi] \; + \; 2\Omega_s^2 \cos(\mu_b) \tau_c \; s \; + \\ &+ \; [\sec^2 \Psi + (\Omega_s \tau_c)^2 \cos \mu_b] s^2 \; + \; 2\tau_c \, s^3 + \tau_c^2 \, s^4 \; . \end{split}$$

Routh determinants

RH(4): tan $\Psi \ge 0$, hence $\Psi \ge 0$. If RH(4) < 0, then the cavity is detuned in the wrong sense.

 $\operatorname{RH}(5)$: $\cos \mu_b \sec^2 \Psi - Y_b \tan \Psi > 0$ implies the Robinson limit: $Y_b < 2 \cos \mu_b / \sin 2\Psi$. If $\operatorname{RH}(5) < 0$, the bunch simply wanders. Substituting the matched generator condition $(\psi_g = 0)$ gives the special case $Y_b < 1 / \sin \mu_b$.

III. CAVITY, BEAM DIPOLE MODE, PHASE-LOOP

The model of section II is supplemented with a beam phase-loop intended to damp bunch dipole oscillations. We assume that the feedback has the response of a pure integrator, and modifies the generator phase ϕ_g , that is $\phi_g = (K_p/s) \times (\phi_b - \phi_V)$. If there is an r.f. feedback around the cavity, this loop modifies the demand phase ϕ_d .

Characteristic polynomial

$$\begin{split} &\Omega_s^2[\cos\mu_b\sec^2\Psi+K_p\tau_c\sin\mu_bY_g\sin\psi_g-Y_b\tan\Psi]+\\ +[K_p\sec^2\Psi+K_pY_b(\sin\mu_b-\cos\mu_b\tan\Psi)+2\Omega_s^2\cos(\mu_b)\tau_c]s+\\ +[\sec^2\Psi+K_p\tau_c(1+Y_b\sin\mu_b)+(\Omega_s\tau_c)^2\cos\mu_b]s^2+2\tau_cs^3+\tau_c^2s^4\\ &\text{A necessary condition for stability is that the coefficient of}\\ s^1\ be\ >\ 0.\ \text{Unless }\tan\Psi\ \leq\ \tan\mu_b\ \text{and}\ K_p\ >\ 0,\ \text{we find a condition for}\ Y_b\ \text{which resembles the Robinson limit;} \end{split}$$

$$Y_b < \frac{2}{\sin 2\Psi} + \frac{2\Omega_s^2 \tau_c}{K_p \tan \Psi} \quad \text{if } \mu_b = 0$$

In most cases this limit is subordinate to RH(5) below.

Routh determinants

$$\begin{split} & \operatorname{RH}(3)\colon 2+K_p\,\tau_c[\cos2\Psi+Y_b\cos\Psi\sin(\Psi+\mu_b)]>0.\\ & \operatorname{This \ condition\ allows\ a\ domain\ of\ stability\ with\ \Psi+\mu_b<0.\\ & \operatorname{The\ damping\ provided\ by\ the\ phase-loop\ can\ overcome\ (partially)\ the\ instability\ caused\ by\ incorrect\ detuning.\\ & \operatorname{RH}(5)\colon\cos\mu_b\sec^2\Psi+K_p\tau_c\sin\mu_bY_g\sin\psi_g-Y_b\tan\Psi>0.\\ & \operatorname{Unless\ }\psi_g\times\mu_b>0\ there\ is\ no\ change\ to\ the\ Robinson\ limit.\\ & \operatorname{RH}(4)\colon\ 0\leq 2K_p\sec^2\Psi[\sec^2\Psi+Y_b(\sin\mu_b-\cos\mu_b\tan\Psi)+\\ &\quad +(\Omega_s\tau_c)^2\cos\mu_b(\cos2\Psi+\tan\mu_b\sin2\Psi)]+\\ & \quad +2K_p(\Omega_s\tau_c)^2\cos\mu_bY_b(\cos\mu_b\tan\Psi-\sin\mu_b)+4\Omega_s^2\tau_cY_b\tan\Psi+\\ &\quad +\tau_cK_p^2[(1+Y_b\sin\mu_b)^2-(Y_g\sin\psi_g\tan\Psi)^2]\ . \end{split}$$

A sufficient condition for RH(4) > 0 is $\tan \Psi = \tan \mu_b$. Alternatively, we may substitute $\psi_g = 0$ and so find RH(4) > 0 at all points on the matched generator curve. Finally, we note that $\mu_b = 0$, $\tan \Psi < 1/\tan \psi_g$ and RH(5) > 0 are sufficient conditions for RH(4) > 0.

IV. CAVITY, BEAM DIPOLE MODE, AND TUNING LOOP

A feedforward (or program) accomplishes the bulk of the cavity tuning. The tuning loop endeavours to bring the generator current and gap voltage vectors in-phase by modifying the cavity resonance frequency. The feedback, for small oscillations about the program set-point, is modelled by a pure integrator: $\tau_c \Delta\Omega_{res} = (K_t/s) \times (\phi_g - \phi_V)$. Since there are no other loops present, $\phi_g = 0$ for all time. The loop will tend to reduce the phase error to zero (i.e. $\phi_g = \phi_V$) provided K_t is positive.

Characteristic polynomial

$$\Omega_s^2 \cos \mu_b K_t (1 - Y_b \sin \mu_b) + 2\tau_c s^4 + \tau_c^2 s^5 + \\ + \Omega_s^2 [\cos \mu_b (\sec^2 \Psi + \tau_c K_t) - Y_b \tan \Psi] s +$$

+ $[K_t + 2\Omega_s^2 \cos(\mu_b)\tau_c]s^2$ + $[\sec^2\Psi + \tau_c K_t + (\Omega_s \tau_c)^2 \cos\mu_b]s^3$. A necessary condition for stability is that the coefficients of s^1 be greater than zero, and this implies

$$Y_b < \cos \mu_b \left[\frac{2}{\sin 2\Psi} + \frac{K_t \tau_c}{\tan \Psi}
ight] \quad \text{if } \Psi > 0$$

However, this condition is subordinate to RH(5).

Routh determinants

 $\operatorname{RH}(3): 2 \sec^2 \Psi + K_t \tau_c \ge 0 .$

RH(4): $K_{\ell}(2 \sec^2 \Psi + K_{\ell} \tau) +$

 $K_t(2 \sec^2 \Psi + K_t \tau_c) + Y_b \Omega_s^2 \tau_c [4 \tan \Psi - K_t \tau_c \sin 2\mu_b] \ge 0$. This condition is usually unimportant for positive detuning $(\Psi > 0)$, and is subordinate to RH(5) for negative detuning. RH(5): This expression can be solved for the beam current Y_b , and is found to factor:

$$Y_b < [0.5K_t \sin 2\mu_b (\sec^2 \Psi + \tau_c K_t) - K_t \tan \Psi + + \Omega_s^2 \cos(\mu_b) \tau_c (2 \tan \Psi - 0.5K_t \tau_c \sin 2\mu_b)] \times (2 \sec^2 \Psi + K_t \tau_c) / \Omega_s^2 \tau_c (2 \tan \Psi - 0.5\tau_c K_t \sin 2\mu_b)^2 .$$

Since the beam current (Y_b) is positive, this leads to a quadratic constraint on the tuning loop gain.

We now simplify the expressions to a non-accelerating beam, to make a correspondence with Reference [4]. In the limit $\mu_b \rightarrow 0$ the stability criterion can be written:

$$Y_b < \left[1 - \frac{K_t}{2\Omega_s^2 \tau_c}\right] \left[\frac{2}{\sin 2\Psi} + \frac{K_t \tau_c}{\tan \Psi}\right]$$

The tuner gain condition, for +ve and -ve tuning angles, can be summarized $(K_t - 2\Omega_s^2 \tau_c) \times \Psi < 0$. The instability regime where $Y_b \ll 1$, $\Psi > 0$ and $K_t > 2\Omega_s^2 \tau_c$ has been experimentally observed in the PSB [4].

V. TUNING LOOP AND BEAM PHASE-LOOP

We supplement the previous model with the ideal phaseloop; $s\phi_g = K_p(\phi_b - \phi_V)$. Because $s\tau_c \Delta\Omega_{res} = K_t(\phi_g - \phi_V)$ there is the possibility for cross-coupling to the tuning loop through the cavity-voltage phase-perturbation ϕ_g .

Characteristic polynomial

$$\Omega_s^2 \cos \mu_b K_t (1 - Y_b \sin \mu_b) + 2\tau_c s^4 + \tau_c^2 s^5 + + \{\Omega_s^2 [\cos \mu_b (\sec^2 \Psi + \tau_c K_t) - Y_b \tan \Psi] + K_p [K_t + \Omega_s^2 \tau_c \sin \mu_b (Y_b \cos \mu_b - \tan \Psi)] \} s + \{K_t + 2\Omega_s^2 \cos \mu_b \tau_c + + K_p [\sec^2 \Psi + \tau_c K_t + Y_b (\sin \mu_b - \cos \mu_b \tan \Psi)] \} s^2 +$$

+[sec²
$$\Psi$$
 + $\tau_c K_t$ + $(\Omega_s \tau_c)^2 \cos \mu_b + \tau_c K_p (1 + Y_b \sin \mu_b)]s^3$.

The coefficients of s^1 and s^2 have the possibility to change sign when $\Psi > 0$. For brevity we give the limit $\mu_b = 0$.

$$Y_b < \frac{2}{\sin 2\Psi} + \frac{K_t(\tau_c + K_p/\Omega_s^2)}{\tan \Psi} \quad \text{when } \mu_b = 0 , \ \Psi > 0 .$$

The coefficient of s^2 is automatically positive if $\tan \Psi \leq \tan \mu_b$; alternatively,

$$Y_b < rac{2}{\sin 2\Psi} + rac{2\Omega_s^2 au_c + K_t (1 + au_c K_p)}{K_p \tan \Psi} \quad ext{if } \mu_b = 0 \,, \ \Psi > 0 \,.$$

<u>Routh determinants</u>

RH(3) factors and simplifies to: $2 + \tau_c K_p \cos 2\Psi$

 $+Y_b \tau_c K_p \cos \Psi \sin(\Psi + \mu_b) + \tau_c K_t (1 - K_p \tau_c) \cos^2 \Psi \ge 0.$

This condition is reminiscent of RH(3) in section III and has the effect of allowing some negative detuning. We should also like RH(3) to be satisfied in the limit $Y_b \rightarrow 0$; and for $K_p \tau_c \gg 1$ this implies the approximate condition: $K_t \tau_c < 2 - \sec^2 \Psi \leq 1$.

RH(4): The Routh determinant has many terms, but simplifies under the substitution $\tan \Psi \Rightarrow Y_b \cos \mu_b$, as occurs when the generator is matched ($\psi_g = 0$); one finds a cubic condition in Y_b . A sufficient stability condition is that the coefficients of Y_b^0 , Y_b^1 , Y_b^2 , Y_b^3 be greater than zero. Only the coefficients of Y_b^0 and Y_b^1 have the possibility to change sign; and so, by inspection, sufficient conditions for RH(4)> 0 are $\tau_c K_t \leq 1$ and $K_p \geq K_t$.

RH(5): The Routh determinant has many decades of monomial terms. Under the condition $\psi_g = 0$, there results a quintic polynomial in Y_b . The condition $\mu_b = 0$ reduces the system to a quadratic in Y_b^2 ; the coefficient of Y_b^4 is unavoidably negative, and so limits the maximum beam current. The allowed domain of Y_b will be maximized when the coefficients of Y_b^0 and Y_b^2 are positive. By inspection, $K_t \tau_c \leq 1$ and $K_p \geq K_t$ is a sufficient condition for both coefficients to be positive.

RH(6): $1 - Y_b \sin \mu_b > 0$ imposes a further constraint on the beam current, which is the same as the no-loop case for a matched generator.

A. R.F. feedback around the cavity

Including a voltage proportional feedback around the cavity modifies the equations. This type of feedback, as discussed in Reference [5], requires a high power summing junction since it is the entire r.f. signal which is fed back. The current \mathbf{I}_g becomes the sum of the demand current \mathbf{I}_d^0 and the feedback current $\mathbf{I}_f = -h\mathbf{I}_V$. It is found that the characteristic polynomials are identical with those of sections II, III, IV, V except with the substitutions: $\tau_c \Rightarrow \tau_c/(1+h)$, $\tan \Psi \Rightarrow \tan \Psi/(1+h)$, $Y_b \Rightarrow Y_b/(1+h)$ made throughout. This being so, we can take over all previous results regarding the polynomial coefficients and Routh-Hurwitz determinants. Generally, the stability limit is enhanced by a factor (1+h).

VI. CAVITY, BEAM DIPOLE AND QUADRUPOLE MODES

Robinson type stability for dipole-quadrupole mode coupling has been investigated in Reference [6], for the case $\mu_b = 0$. We generalize to the case of an accelerating beam.

A. Rigid bunch quadrupole motion

Let bunch half-length be $\Theta = \Theta_0 + \theta$, the sum of a steady state part Θ_0 and a small perturbation $\theta(t)$. The Laplace transform of the envelope oscillation can be derived from: $s \theta = \Omega_s^2 \delta W$ and $s \delta W = -4 \cos \mu_b \times \theta - z_V \Theta_0 \cos \mu_b$. where the variable δW is conjugate to θ . To complete our description of the beam coupling to the cavity, we give the relation between θ and amplitude modulation of the beam current z_b . To first order $z_b + F_0 \times \theta = 0$. The form factor F_0 depends on the bunch shape, λ . Let J_n be Bessel functions. For the functions $\lambda = (\Theta_0^2 - x^2)^{\alpha}$ with $\alpha > 0$,

$$F_0(\Theta_0) = (2\alpha + 1)/\Theta_0 - J_{\alpha - 1/2}(\Theta_0)/J_{\alpha + 1/2}(\Theta_0) .$$

For example, if $\alpha = 1$ then $F_0 \approx \Theta_0 / 5$ when $\Theta_0 < 1$.

Characteristic polynomial

The polynomial is too lengthy to reproduce here. We consider $\mu_b > 0$, in which case only the coefficient of s^2 has the possibility to change sign when $\Psi > 0$; this implies a beam current limit, but the condition is subordinate to those below.

<u>Routh determinants</u>

RH(3): $2 \sec^2 \Psi - Y_b F_0 \Theta_0 (\Omega_s \tau_c)^2 \cos \mu_b \sin \mu_b > 0$. This constraint is quite severe for small tuning angles and long bunches, but is subordinate to RH(6).

If RH(3)>0 then a sufficient condition for RH(4)>0 is: $\tan \Psi \ge \sin 2\mu_b F_0 \Theta_0 (1+2\Omega_s^2 \tau_c^2 \cos \mu_b)/2(1+F_0 \Theta_0 \cos^2 \mu_b)$. RH(5) simplifies very slightly to a condition with 24 monomial terms, and there is no simple interpretation. In the limit of large tuning angle, short bunch length, and $\Omega_s \tau_c$

order of or less than unity, we find the approximation: $2 \tan \Psi [2 \cos \mu_b - Y_b \sin 2\Psi] + F_0 \Theta_0 \cos \mu_b [16 \cos^2 \mu_b \tan \Psi + 4 \sin 2\mu_b \sec^2 \Psi + 2Y_b (2 \cos \mu_b - \sin \mu_b \tan \Psi - 4 \sin^2 \Psi)] > 0.$ The leading term in tan Ψ contains the Robinson limit.

RH(6) factors; if RH(3) > 0 and RH(4) > 0 this leaves the new condition $Y_b < 3 \tan \Psi / [F_0 \Theta_0 \cos \mu_b]$ which poses a severe constraint at small tuning angles unless μ_b is large or the bunches are short.

RH(7): $4(\cos \mu_b \sec^2 \Psi - Y_b \tan \Psi) + Y_b F_0 \Theta_0 \cos \mu_b (\sin \mu_b - \cos \mu_b \tan \Psi + Y_b) > 0$. The term in Y_b^2 in this quadratic will favourably modify the stability compared with the Robinson limit. However, for small tuning angles condition RH(6) supersedes RH(7).

VII. REFERENCES

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