PERTURBATION THEORY OF BROADBAND IMPEDANCES*

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Abstract

A perturbation theory for broadband impedance calculations has been developed, allowing evaluation of impedances for an accelerating structure of a rather arbitrary shape. General formulas are given for the longitudinal and transverse impedances. The method is checked by calculating impedances and comparing results with those for structures previously studied. Several new results, including impedance of a taper, are presented.

I. INTRODUCTION

The interaction of a beam with the beam environment in accelerators is usually described in terms of the coupling impedances, with most of the impedance calculations performed using numeric codes. This paper describes a perturbation theory for the impedance calculations based on Kirchhoff's equations, analogous to the Born series in the scattering theory. A perturbation theory of this kind was used in the time domain by Novokhatsky [1], and by the author [2] for more general structures in the frequency domain. A cylindrical symmetry is implied in most cases, unless it is stated otherwise, although the method also may be applied to study impedances of structures without cylindrical symmetry. ring.

II. ILLUSTRATION: THE METHOD

Consider a well known electrostatic problem: find the field of a point-like charge e placed at distance $z = a$ from an ideal conducting x, y plane. The field potential for $z > 0$ is a superposition of the potential $\phi_{\rm ext}$ of a charge in free space and the potential of the image charge $-e$ at $z = -a$. This result may be obtained using Green theorem [3] volume:

$$\phi(\vec{R}) \;=\; \phi_{\rm ext}(\vec{R}) + \int \frac{d\vec{S}'}{4\pi} \times [G(\vec{R},\vec{R}')\,\vec{\nabla}'\,\phi(\vec{R}') - \phi(\vec{R}')\,\nabla'\,G(\vec{R},\vec{R}')] \;. \tag{1}$$

Solve (1) by iterations: $\phi = \phi^{(0)} + \phi^{(1)} + \ldots$. In the zeros approximation, $\phi^{(0)} = \phi_{\rm ext}$. In the nth approximation

$$\phi(\vec{R}) \;=\; \phi_{\rm ext}(\vec{R}) - e/[(z+a)^2 + r^2]^{1/2} \times [\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 \ldots] \,, \tag{2}$$

giving the correct answer.

Note that although the final result satisfies the boundary condition, the result of any finite number of iterations does not. Hence, the solution is exact for each iteration, but the boundary conditions are satisfied only approximately.

The perturbation method based on Kirchhoff's integral equation gives [4] the impedances for the monopole and dipole modes

$$Z_l^0(k) \;=\; -\,\frac{ikZ_0}{2\pi} \int dz a'(z) \int dz' a'(z') \int d\phi' \times \left[G_k(\vec{R},\,\vec{R}')\right]_{r=a(z),\,r'=a(z')} \cos(\phi-\phi')\, e^{-ik(z-z')} \;. \tag{3}$$

$$Z_l^{(1)}(k) \;=\; -\,Z_0\, \frac{ikrr_0\cos\phi}{2\pi} \int d\phi \cos 2(\phi-\phi') \int dz dz' \times e^{-ik(z-z')} \left\{ \frac{a'(z)\,a'(z')}{a(z)\,a(z')} \right\} [G_k(\vec{R},\,\vec{R}')]_{r=a(z),\,r'=a(z')} \,. \tag{4}$$

The transverse impedance then is given by the Wenzel-Panofsky theorem.

Equations (3) and (4) give a close form of the longitudinal and transverse impedances for a cylindrically symmetric beam pipe, with an arbitrary variation of the pipe radius $a(z)$. From these equations it is also easy to obtain the longitudinal and transverse wakefields.

III. EXAMPLES OF LONGITUDINAL IMPEDANCE

For a hole in a beam pipe, the imaginary part of the impedance

$$ImZ_l(k) \;=\; \frac{w^2}{(2\pi)^2 a^2 c} \int_0^{2kL} \frac{dx}{x} \sin(x) \;. \tag{5}$$

If the slot is short $kL \ll 1$, then

$$ImZ_l(k) \;=\; Z_0\, \frac{kLw^2}{(2\pi)^3\, a^2} \tag{6}$$

reproduces the Kurennoy's result [4]. The impedance increases with L for short slots $kL \ll 1$, and goes to a constant for $kL \gg 1$.

For a shallow cavity $(b-a) \ll a$, $g \ll a$, $k\,[g^2 + (b-a)^2]^{1/2} \ll 1$ the longitudinal impedance obtained from simulations with the code TBCI for long bunches is inductive [5]. We obtain for this case

$$L \;=\; \frac{Z_0(b-a)^2}{(2\pi)^2\, a}\, f(\lambda) \,, \tag{7}$$

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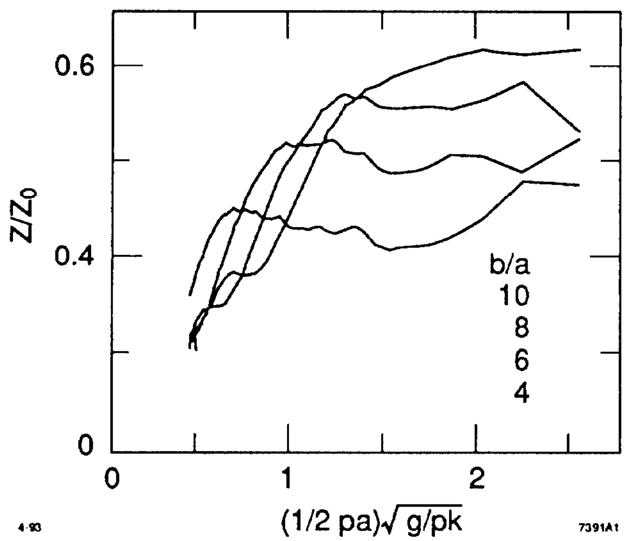

Figure 1. The real part of the longitudinal impedance of a cavity as a function of the Dome-Lawson parameter: $g/a = 3.0$, b/a in the range $2.0 - 6.0$. The transition from the regime of a cavity to the regime of a step is shown.

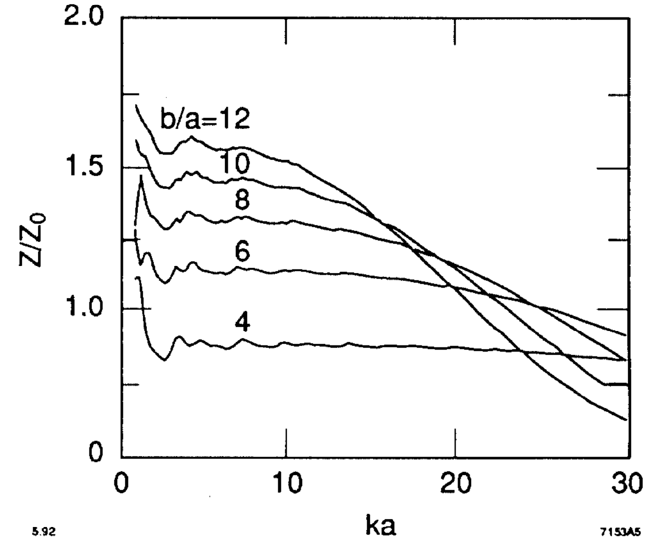

Figure 2. The frequency dependence of the impedance of a collimator. The impedance is constant for small ka and rolls off at large frequencies. The roll-off starts at frequencies dependent on the ratio of the radii.

where $\lambda = g/(b-a)$ and $f(\lambda) \approx 1$. For small $\lambda \ll 1$ it gives K. Bane's result.

For a shallow collimator the inductance

$$L = \frac{Z_0(b-a)^2}{4\pi a} \ln\left[\frac{2\pi a}{(b-a)}\right] + (3/2) , \qquad (8)$$

which is similar to the TBCI result

$$L = \frac{Z_0(b-a)^2}{(\pi a)} .$$

Impedance of a cavity in the high-frequency limit $kg \gg 1$, $ka \gg 1$. has been studied before [7]. The real part of the impedance is

$$\mathrm{Re} Z_l = \frac{Z_0}{2\pi a} \left(\frac{g}{\pi k}\right)^{1/2} . \qquad (9)$$

For this geometry, the general formulathe expression for the real part of the impedanceis given by the interval $-k < p < k$:

$$\mathrm{Re}\ Z_l^{(0)}(k) = \frac{kZ_0}{2\pi} \int_{-k}^{k} \frac{dp}{(k^2-p^2)} \sin^2$$
$$\times \frac{g}{2}(k-p)\left[J_0(\Omega a) - J_0(\Omega b)\right]^2 . \qquad (10)$$

At high frequencies, $ka \gg 1$, $kb \gg 1$, we obtain the Dome-Lawson result (9).

For very large gaps g, the impedance does not depend on g, but depends on both radii. Transition from the regime of a cavity to the regime of a step occurs [7] at $g \simeq k(b-a)^2$.

The result of the numerical integration of Eq. (10) is shown in Fig. 1.

The impedance of a collimator can be derived similarly to the impedance of a cavity. The impedance calculated from this formula is shown in Fig. 2.

The radius of a taper varies linearly from a to $b > a$ at distance L. The longitudinal impedance is

$$Z_l^{(0)}(k) = \frac{Z_0}{\pi} \ln\left(\frac{b}{a}\right) + \frac{kZ_0(a')^2}{8\pi} S(k) , \qquad (11)$$

where

$$S(k) = \int dp \int_0^L dz dz' \exp\{i(p-k)(z-z')\}$$
$$\times \left[G_{1,p}(z,z')\right]\Big|_{r=a(z),\, r'=a(z')} . \qquad (12)$$

The integral (12 can further be reduced to a single integral.

Results of the numerical integration of Eq. (12) are shown in Figs. 3, 4, and 5.

IV. CONCLUSION

The perturbation method described above reproduces numerous previously known analytical results. This method allows us to obtain all these results in a unified way as extreme cases of the same formula, and to demonstrate the transition from one case to another; for example, from the regime of a cavity to the regime of a step, or from a single cavity to a periodic array. The method can be generalized to more complicated geometries.

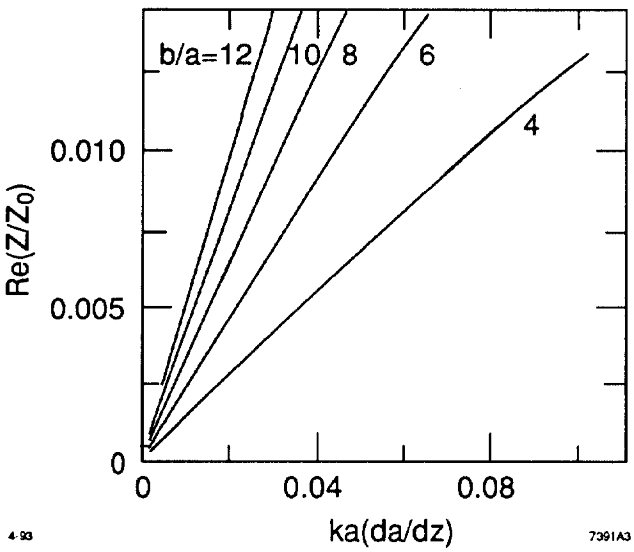

Figure 3. The real part of the longitudinal impedance of a taper, with large $p = b/a$.

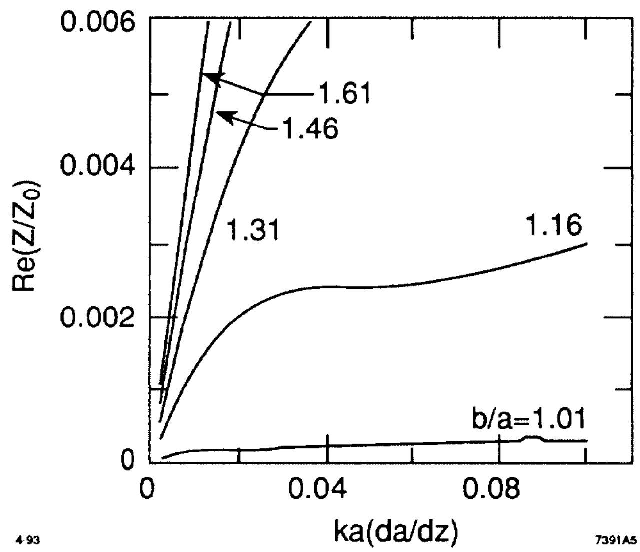

Figure 5. The transition from $(p-1) \ll 1$ to $(p-1) \simeq 1$.

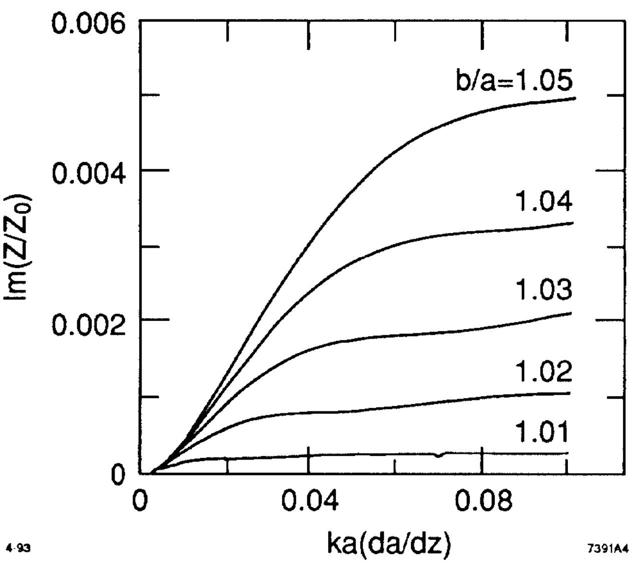

Figure 4. The real part of the longitudinal impedance of a taper, for small $(p-1) \ll 1$.

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