A Generalized Method for Calculating Wake Potentials

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Abstract

We describe a generalized method to compute wake potentials created in axisymmetric structures. It relies on expressing the wake potentials, of any multipole order, as integrals over the e.m. fields along an arbitrary one-dimensional contour spanning the structure longitudinally. For perfectly conducting structures, the integration along the axis can then be replaced by choosing a contour beginning and ending on the beam tubes. Thus it generalizes the former method of calculating the wake potentials by integrating along a straight line at the beam tube radius. Its usefulness is illustrated with the computer code ABCI which permits calculation of wake potentials in structures extending to the inside of the beam tube radius, or having unequal beam tube radii at the two sides.

I. INTRODUCTION

The determination of the wake potentials and impedances created by metallic structures surrounding the beam trajectory is an important issue in the design of accelerators. In most practical cases, the wake fields must be calculated with computer codes. For cavity-like structures symmetric about the beam axis, using the known radial dependence of the monopolar (m=0) longitudinal and dipolar (m=1) transverse potentials, the integration of the wake fields can be performed along a straight line parallel to the axis at the beam tube radius [1]. For perfectly conducting walls, the boundary conditions ensure that the integral along the beam tube vanishes for the tangential (longitudinal or azimuthal) components of the wake potential. This leaves the integral across the cavity gap as the only contribution to the wake potentials. This simplification is essential for computer calculations, in particular for long structures and short bunches requiring small mesh size and where long beam tubes would require excessive computer memory and cpu time. However, this technique does not work when the two beam tubes have unequal radii, or when part of the cavity extends to a smaller radius. If it is the case, for instance for tapers, steps, collimators or cavities with small aperture irises, the only alternative is to integrate along a straight line at an allowable radius, and with beam tubes as long as possible. Usually one must also subtract the wake potential of the beam tubes without structure ("numerical noise") which is different from zero due to the discretization of the geometry.

In this paper, we generalize the above straight-line integration method, by showing that the longitudinal and transverse wake potentials, at all orders m in the multipolar expansion, are given by a wakefield integral along any arbitrary contour, like (C) in Figure 1, starting and ending on the beam tubes. This integral is such that the contribution of the beam tubes vanishes. One can therefore treat more general structures by passing underneath the lowest radius material without having to introduce long beam tubes. The detailed derivation of this method is given in [2]. Its only limitation is, for $m \geq 1$, that the two beam tubes have equal radii.

This method has been implemented in the computer code ABCI [3] where the integration path is made of straight line segments defined by the 3 parameters ZCF, ZCT ans RWAK as shown in Figure 1. Results from this code are presented.

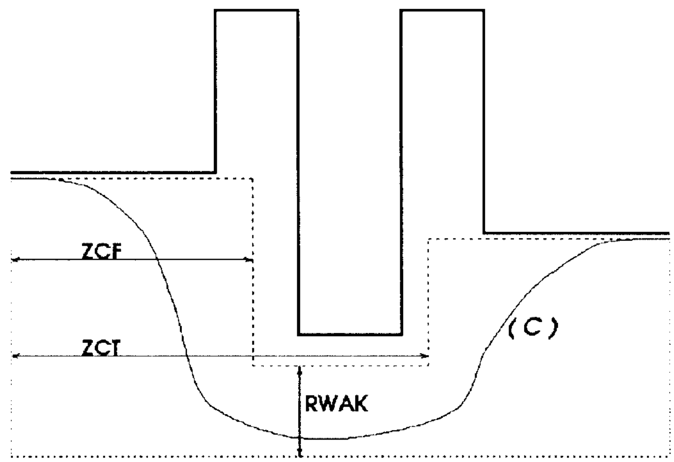

Figure 1 Contours and contour parameters in program ABCI

II. CALCULATION OF THE WAKE POTENTIALS

The longitudinal and transverse wake potentials are defined by

$$W_z(r,\theta,s) = -\frac{1}{Q}\int_{-\infty}^{+\infty} dz\, E_z(r,\theta,z,t(z,s)) \tag{1}$$

and

$$\mathbf{W}_\perp(\mathrm{r},\theta,\mathrm{s}) = \frac{1}{Q}\int_{-\infty}^{+\infty} dz\,(\mathbf{E}_\perp+\mathbf{v}\times\mathbf{B})(r,\theta,z,t(z,s)) \tag{2}$$

where s is the distance behind the exciting charge Q of velocity v=c.

$$t(z,s) = (z+s)/c \ . \tag{3}$$

It is convenient to decompose the electromagnetic fields as $\mathbf{E}=\mathbf{E}^{(0)}+\mathbf{E}^{(r)}$ and $\mathbf{B}=\mathbf{B}^{(0)}+\mathbf{B}^{(r)}$ where $\left(\mathbf{E}^{(0)},\mathbf{B}^{(0)}\right)$ are the

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fields generated by Q in free space, and $\left(\mathbf{E}^{(r)}, \mathbf{B}^{(r)}\right)$ are the fields radiated by the structure and contributing to the wake potentials W_z and $\mathbf{W}_\perp$. Assuming $\theta = 0$ for the exciting charge, these fields obey the following Fourier expansion

$$
\begin{aligned}
(E_r, B_\theta, E_z)(r,\theta,z,t) &= \sum_{m=0}^{\infty} (e_r, b_\theta, e_z)^{(m)}(r,z,t)\cos(m\theta) \\
(B_r, E_\theta, B_z)(r,\theta,z,t) &= \sum_{m=1}^{\infty} (b_r, e_\theta, b_z)^{(m)}(r,z,t)\sin(m\theta)
\end{aligned}
\tag{4}
$$

Defining the 2d-vectors $\mathbf{S}^{(m)}$ and $\mathbf{D}^{(m)}$ in the *(r,z)*-plane, as

$$
\mathbf{S}^{(m)} = \begin{pmatrix} r^m \left[e_r^{(r)} + cb_\theta^{(r)} - e_\theta^{(r)} + cb_r^{(r)}\right]^{(m)} \\ r^m \left[e_z^{(r)} + cb_z^{(r)}\right]^{(m)} \end{pmatrix} \tag{5}
$$

$$
\mathbf{D}^{(m)} = \begin{pmatrix} r^{-m} \left[e_r^{(r)} + cb_\theta^{(r)} + e_\theta^{(r)} - cb_r^{(r)}\right]^{(m)} \\ r^{-m} \left[e_z^{(r)} - cb_z^{(r)}\right]^{(m)} \end{pmatrix} \tag{6}
$$

one can show, using the homogeneous Maxwell's equations satisfied by the fields $\left(\mathbf{E}^{(r)}, \mathbf{B}^{(r)}\right)$, that the one-forms defined by these vectors are closed in the *(r,z)*-plane, i.e.

$$
\begin{aligned}
\partial_r S_z^{(m)}(r,z,s,t(z,s)) - \partial_z S_r^{(m)}(r,z,s,t(z,s)) &= 0 \\
\partial_r D_z^{(m)}(r,z,s,t(z,s)) - \partial_z D_r^{(m)}(r,z,s,t(z,s)) &= 0
\end{aligned}
\tag{7}
$$

This implies that the vectors $\mathbf{S}^{(m)}$ and $\mathbf{D}^{(m)}$ derive from a potential and that their integral along a closed contour (enclosing the vacuum) vanishes. This property allows one to deform the wakefield integration path from the straight line (L_r) at constant radius r used in definitions (1) and (2), to any contour (C).

A. The m=0 case

In this case the radiated fields $\left(\mathbf{E}^{(r)}, \mathbf{B}^{(r)}\right)$ vanish at both ends of the beam tube, and the integrals of $\mathbf{S}^{(0)}$ along (L_r) and along the path (C), are equal:

$$
\int_{L_r} \mathbf{S}^{(0)}(r,z,s)\cdot d\mathbf{l} = \int_{C} \mathbf{S}^{(0)}(r',z,s)\cdot d\mathbf{l} \tag{8}
$$

The right-hand side of this equation is proportional to the longitudinal potential $W_z^{(0)}(s)$, which is therefore given by

$$
\begin{aligned}
W_z^{(0)}(s) = &-\frac{1}{Q}\int_C \left[E_z dz + (E_r + cB_\theta)dr\right]^{(m=0)}(r,z,t(z,s)) \\
&+ \frac{1}{\pi\epsilon_0}\ln\left[\frac{a_{in}}{a_{out}}\right]\delta(s)
\end{aligned}
\tag{9}
$$

where the log term comes from the integration of the free fields $E_r^{(0)} + cB_\theta^{(0)}$, and a_{in} and a_{out} are the tube radii.

B. The m>0 case

In that case we assume that $a_{in} = a_{out} = a$, as will be justified later. Then

$$
\int_{L_r} \mathbf{S}^{(m)}(r,z,s)\cdot d\mathbf{l} = \int_{C} \mathbf{S}^{(m)}(r',z,s)\cdot d\mathbf{l} \tag{10}
$$

$$
\int_{L_r} \mathbf{D}^{(m)}(r,z,s)\cdot d\mathbf{l} = \int_{C} \mathbf{D}^{(m)}(r',z,s)\cdot d\mathbf{l} \tag{11}
$$

since the radial integrals at $z = \pm\infty$ cancel each other. The first equation, evaluated on the z-axis, implies that the integral of $\mathbf{S}^{(m)}$ along any contour vanishes. It is then easy to show that

$$
W_z^{(m)}(r,\theta,s) = -\frac{\cos(m\theta)}{2Q} r^m \int_C \mathbf{D}^{(m)}\cdot d\mathbf{l} \tag{12}
$$

Adding to the integral of $\mathbf{D}^{(m)}$ the (vanishing) integral of $\mathbf{S}^{(m)}$ along (C) divided by a^{2m}, leads to an expression of the longitudinal wake potential where the integral along the beam tubes vanishes due to the metallic boundary conditions. This is however only possible when the tube radii are equal. Evaluating the resulting expression in terms of the e.m. fields in the structure, leads to

$$
W_z^{(m)}(r,\theta,s) = \frac{r^m\cos(m\theta)}{2Qa^m} w^{(m)\prime}(s) \tag{13}
$$

with

$$
\begin{aligned}
w^{(m)\prime}(s) = -\int_C & dz\left[\left(\frac{a^m}{r'^m} + \frac{r'^m}{a^m}\right)e_z - \left(\frac{a^m}{r'^m} - \frac{r'^m}{a^m}\right)cb_z\right] \\
&+ dr'\left[\left(\frac{a^m}{r'^m} + \frac{r'^m}{a^m}\right)(e_r + cb_\theta) + \left(\frac{a^m}{r'^m} - \frac{r'^m}{a^m}\right)(e_\theta - cb_r)\right]
\end{aligned}
\tag{14}
$$

The transverse potential can be written, using the Panofsky-Wenzel theorem, as

$$
\mathbf{W}_\perp(r,\theta,s) = \sum_{m=1}^{\infty} \frac{mr^{m-1}}{2Qa^m}\left(\cos(m\theta)\hat{\mathbf{r}} - \sin(m\theta)\hat{\Theta}\right) w^{(m)}(s) \tag{15}
$$

with

$$
\begin{aligned}
w^{(m)}(s) = \frac{-1}{m}\int_C & r' dr'\left[\left(\frac{a^m}{r'^m} - \frac{r'^m}{a^m}\right)e_z - \left(\frac{a^m}{r'^m} + \frac{r'^m}{a^m}\right)cb_z\right] \\
&+ r' dz\left[\left(\frac{a^m}{r'^m} + \frac{r'^m}{a^m}\right)(e_\theta + cb_r) - \left(\frac{a^m}{r'^m} - \frac{r'^m}{a^m}\right)(e_r - cb_\theta)\right]
\end{aligned}
\tag{16}
$$

In equations (14) and (16), it is understood that the electromagnetic fields are projected on their multipolar component of order m, and that their argument is $(r', z, t(z,s))$.

III. COMPUTER IMPLEMENTATION

The possibility of integrating along a non-straight path, using Equations (9), (14) and (16), has been implemented in the time-domain program ABCI [3] as discussed in the introduction. We illustrate the interest of this method with the calculation of wake potentials for two cases. We first consider a 1 cm long collimator of 4 mm radius in a beam tube of 1 cm radius. Figure 2 shows a comparison of the loss factor of a Gaussian bunch with $\sigma_z = 5\,\mathrm{mm}$ calculated with two different methods:

1. the wakefield integration along a straight line at 4 mm constant radius (solid line). The calculated wake potential and loss factor then depend on the length L of the tube on both sides of the collimator. The result is given, after substraction of the wake of the tube alone ("numerical noise correction" similar ot the WAKCOR method in TBCI [5]), by the asymptotic value.
2. the wakefield integration along the boundary of the collimator, using Equation (9). In this case the result is independent of the length of the tube and gives directly the value of the loss factor (dotted line).

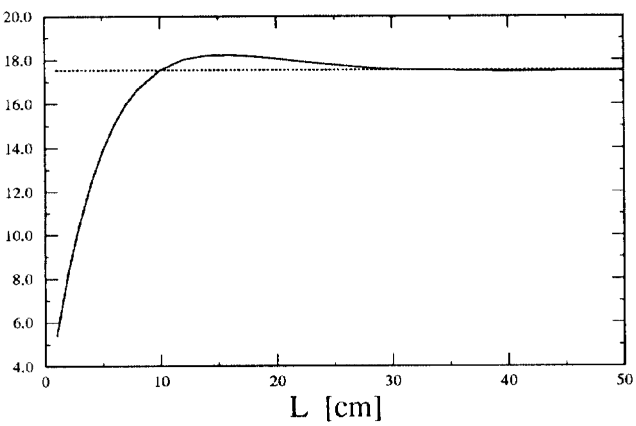

Figure 2 Longitudinal loss factor [V/pC] of a collimator as a function of the beam tube length

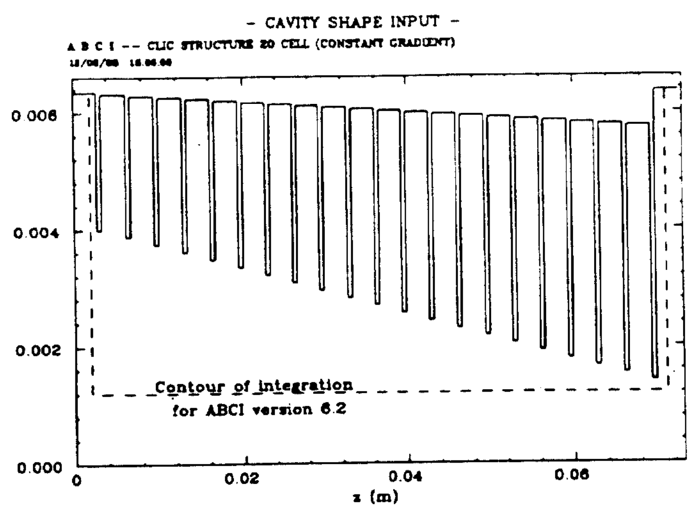

Figure 3 Constant gradient structure for CLIC

Finally Figure 4 plots the m=1 wake potentials of a 20 cell 30 GHz constant gradient structure, as shown in Figure 3, where the inner and outer radii of each cell are different. The contour of integration chosen by the program is given by the dashed line.

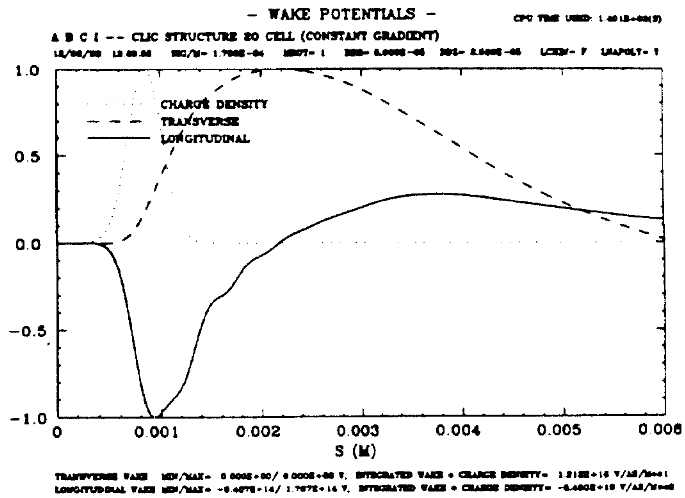

Figure 4 Dipolar wake potential (m=1)

IV. CONCLUSION

In practical calculations of the wake potentials created by axisymmetric cavities, one usually evaluates them by integrating along the cavity gap at the beam tube radius. We have generalized this method by showing that the wake potentials, of any multipole order, are given by integrals over the wake fields along any arbitrary contour spanning the structure longitudinally. By so doing we have extended the range of applications to structures of more complicated shape. The integration of wake fields along well chosen contour permits a large savings in computer capacity. In particular, the integration along a structure extending to the inside of the beam tubes - such as a collimator or iris - has become much easier with this method. Also the m=0 wake potential of structures with unequal beam tubes can be calculated in this manner. The new method of integration has been implemented in the code ABCI (versions 5 or higher) which can choose the proper contour automatically or as selected by the user.

V. REFERENCES

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