Resistive Wall Wake Function for Arbitrary Pipe Cross Section

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Abstract

A method for calculating the impedance and the wake function of resistive beam pipes is given. It allows an arbitrary shape of the crosssection, and arbitrary location of the source and witness particles, including the short-range behavior of the wake function. The pipe is uniform longitudinally and the beam is assumed to be ultra-relativistic. A simple computer code has been written using the boundary element method. Some results for elliptical, rectangular and hyperbolic pipes are presented.

I. METHOD OF COMPUTATION

The resistive wall impedance has been a topic in accelerator physics since 1960's and is thought to be well known but, in fact, it calls for further investigation still now as the recent papers show [1,2,3,4,5]. The aim of the present paper is to give a method of computing the impedance/wake function for arbitrary shape of the beam pipe, arbitrary location of the particles including the short range behavior of the wake. The detail of our method is described in [6].

A. Assumptions and notation

We assume a longitudinally uniform pipe and an ultra-relativistic beam. The z-axis is parallel to the pipe. In the (x,y) plane, the vacuum region surrounded by the wall is denoted by Ω. The coordinate s is the length measured along the wall surface $\partial\Omega$. The unit normal (outward from Ω) and tangential vectors at s are denoted by $\boldsymbol{n}(s)$ and $\boldsymbol{\tau}(s)$, respectively, and the unit vector along z by $\boldsymbol{e}_z$. The source and the witness charge (unit charge) are located at $\boldsymbol{r}_s = (x_s, y_s)$ and $\boldsymbol{r}_w = (x_w, y_w)$, respectively. All the field quantities are proportional to $\exp i(kz - \omega t)$ $(k = \omega/c)$ because of the longitudinal uniformity.

We write the electric and magnetic fields as $\boldsymbol{E}+\boldsymbol{E}^{(0)}$ and $\boldsymbol{H}+\boldsymbol{H}^{(0)}$, where the superscript (0) denotes solutions for the perfectly conducting wall with the same wall shape. Since the transverse Lorentz force $\boldsymbol{F}_\perp = \boldsymbol{E}_\perp + Z_0\boldsymbol{e}_z\times\boldsymbol{H}$ satisfies $\boldsymbol{F}_\perp = -i/k\nabla_\perp E_z$, all the needed information is obtained from E_z.

B. Kirchhoff integral formula

Since $(\boldsymbol{E},\boldsymbol{H})$ obeys the sourceless Maxwell equation, it satisfies the Kirchhoff integral formula, which, when the field is proportional to $e^{ik(z-ct)}$, can be written as

$$E_z(\boldsymbol{r}) = \oint ds' \left[ik(E_n - Z_0H_\tau)G - E_z\boldsymbol{n}'\cdot\nabla'_\perp G\right], \quad (1)$$

$$\boldsymbol{E}_\perp(\boldsymbol{r}) = \oint ds'[ik(\boldsymbol{\tau}'Z_0H_z - \boldsymbol{n}'E_z)G - E_n\nabla'_\perp G - E_\tau\boldsymbol{e}_z\times\nabla'_\perp G], \quad (2)$$

Here, $\oint$ is the integral along $\partial\Omega$ and the prime refers to the quantities evaluated on the wall at s'. The function $G = G(\boldsymbol{r},\boldsymbol{r}')$ is the Green function satisfying the Laplace equation $\Delta_\perp G(\boldsymbol{r},\boldsymbol{r}') = -\delta(\boldsymbol{r}-\boldsymbol{r}')$. The simplest choice is $G(\boldsymbol{r},\boldsymbol{r}') = -(1/2\pi)\log|\boldsymbol{r}-\boldsymbol{r}'|$.

The limit that $\boldsymbol{r}\to\partial\Omega$ gives integral equations where only the fields on the wall appear. For numerical calculation, we divide the wall surface $\partial\Omega$ into short segments and express the field as column vectors $|\,\rangle$. Then, the integrals appearing in eqs.(1) and (2) can be represented by matrices. Thus, we obtain matrix equation

$$\mathcal{D}\,|E_z\rangle = -ik\mathcal{G}\,|Z_0H_\tau - E_n\rangle\,, \quad (3)$$

$$\mathcal{N}\,|E_n\rangle - \mathcal{T}\,|E_\tau\rangle = -ik\left[\mathcal{C}\,|E_z\rangle + \mathcal{S}\,|Z_0H_z\rangle\right], \quad (4)$$

$$\mathcal{T}\,|E_n\rangle + \mathcal{N}\,|E_\tau\rangle = -ik\left[\mathcal{S}\,|E_z\rangle - \mathcal{C}\,|Z_0H_z\rangle\right]. \quad (5)$$

C. Approximate boundary condition

When the skin depth is much smaller than the typical transverse scale L of the pipe, the boundary condition is:

$$Z_0(H_\tau + H_\tau^{(0)}) = -\frac{\kappa}{k}E_z, \quad Z_0H_z = \frac{\kappa}{k}E_\tau. \quad (\text{on } \partial\Omega) \quad (6)$$

$$\kappa \equiv e^{\pi i/4}\sqrt{\mu_0 ck\sigma} = (1+i)/\delta_{skin} = e^{\pi i/4}\sqrt{k/\rho_0}, \quad (7)$$

$$\delta_{skin} = \sqrt{2/\mu_0 ck\sigma}, \qquad \rho_0 = 1/\mu_0 c\sigma \quad (8)$$

where μ_0 is the permeability of vacuum and σ the conductivity of the wall material. ($\rho_0 \approx 0.5\times10^{-10}$m for copper at room temparature.)

D. Solution of the integral equation

The E_τ terms on the l.h.s. of (4) and (5) can be ignored when $\delta_{skin} \ll L$. Then, eqs.(4) and (5) can formally be solved as

$$|E_n\rangle = -ik\mathcal{M}\,|E_z\rangle\,, \quad (9)$$

where $\mathcal{M}$ is a matrix defined by

$$\mathcal{M} = (\mathcal{N} + \mathcal{S}\mathcal{C}^{-1}\mathcal{T})^{-1}(\mathcal{C} + \mathcal{S}\mathcal{C}^{-1}\mathcal{S}). \quad (10)$$

Using eq.(9) and the boundary condition (6), we can solve eq.(3), under the same condition $\delta_{skin} \ll L$, as

$$|E_z\rangle = \frac{k}{\kappa}\left[1 - i\frac{k^2}{\kappa}\mathcal{M}\right]^{-1}\left|Z_0H_\tau^{(0)}\right\rangle. \quad (11)$$

The solution $Z_0H_\tau^{(0)}$ for the perfectly conducting case, appearing on r.h.s. of this equation, can be found using the matrices $\mathcal{G}$ and $\mathcal{D}$ defined in (3) as

$$|u\rangle \equiv \left|Z_0H_\tau^{(0)}\right\rangle = \mathcal{G}^{-1}\mathcal{D}\,|g_s\rangle - \left|\frac{\partial g_s}{\partial n}\right\rangle, \quad g_s(\boldsymbol{r}) = G(\boldsymbol{r},\boldsymbol{r}_s) \quad (12)$$

E. The field at the witness particle

To find E_z at the witness particle, we go back to the Kirchhoff formula (1), which can formally be written as

$$E_z(\boldsymbol{r}_w) = -ik\,\langle g_w|Z_0H_\tau - E_n\rangle - \langle g'_w|E_z\rangle\,, \quad (13)$$

where g_w and g'_w are functions on $\partial\Omega$ defined by

$$g_w(s) = G(\boldsymbol{r}_w, \boldsymbol{r}) \quad \text{and} \quad g'_w = \partial g_w/\partial n = \boldsymbol{n}\cdot\nabla_\perp G(\boldsymbol{r}_w, \boldsymbol{r}).$$

The brakets $\langle\ |\ \rangle$ denote the integral over the circumference of the cross section: $\langle u|v\rangle \equiv \oint ds\, u^*(s)\, v(s)$. Using eq.(3), we can rewrite eq.(13) as $E_z(\boldsymbol{r}_w) = \langle v|E_z\rangle$ where $\langle v| \equiv \langle g_w|\mathcal{G}^{-1}\mathcal{D} - \langle g'_w|$.

Thus, finally we find E_z at the witness particle

$$E_z(\boldsymbol{r}_w) = Z_0\frac{k}{\kappa}\langle v|\left[1 - i\frac{k^2}{\kappa}\mathcal{M}\right]^{-1}|u\rangle. \tag{14}$$

F. Eigenvalues and eigenfunctions of the matrix $\mathcal{M}$

We denote the eigenvalue (always real) of $\mathcal{M}$ by μ_α and the eigenfunction by $|\alpha\rangle$, which is normalized as $\langle\alpha|\alpha\rangle = 1$. Then, eq.(14) can be written as

$$E_z(\boldsymbol{r}_w) = Z_0\frac{k}{\kappa}\sum_\alpha \frac{c_\alpha}{1 - i(k^2/\kappa)\mu_\alpha}, \tag{15}$$

with $c_\alpha(\boldsymbol{r}_w, \boldsymbol{r}_s) = \langle v|\alpha\rangle\langle\alpha|u\rangle$. The transverse force is then given by

$$\boldsymbol{F}_\perp(\boldsymbol{r}_w) = -i\frac{Z_0}{\kappa}\sum_\alpha \frac{\partial c_\alpha/\partial\boldsymbol{r}_w}{1 - i(k^2/\kappa)\mu_\alpha}, \tag{16}$$

where the coefficients $\partial c_\alpha/\partial\boldsymbol{r}_w$ can be calculated simply by using $\partial g_w/\partial\boldsymbol{r}_w$ instead of g_w. From eqs.(15) and (16), we can calculate any physical quantities like the wake function, loss parameters, wall heating etc. Note that $\mathcal{M}$ depends only on the wall shape but is independent of the particle location, the frequency, and the conductivity.

G. Asymptotic form for $k \ll (L^2\rho_0)^{-1/3}$

In the asymptotic region $k \ll (L^2\rho_0)^{-1/3}$, μ_α in eqs.(15) and (16) can be ignored. Since $\sum_\alpha |\alpha\rangle\langle\alpha|$ is identity, we get

$$E_z = \frac{Z_0 k}{\kappa}\langle v|u\rangle = \frac{Z_0 k}{\kappa}\oint ds\, v^*(s)u(s), \quad (k \ll (L^2\rho_0)^{-1/3}). \tag{17}$$

When $\boldsymbol{r}_s = \boldsymbol{r}_w$, we have $u = v$ and, consequently, the integrand becomes $|u(s)|^2$, giving rise to a formula identical to the longitudinal impedance formula obtained in [5]. Similar formulas can be found for the transverse impedance. Thus, if one is interested only in the asymptotic form, the operator $\mathcal{M}$ is not needed.

H. Wake function

Since all the terms in eqs.(15) and (16) have the same wave number dependence, the wake function can easily be computed from two basic functions f_L and f_T

$$W_L(z) = \sum_\alpha \frac{cZ_0}{\mu_\alpha} c_\alpha f_L(z/z_\alpha), \quad z_\alpha = [(2\mu_\alpha)^2\rho_0]^{1/3} \tag{18}$$

$$\boldsymbol{W}_\perp(z) = \sum_\alpha \frac{cZ_0 z_\alpha}{\mu_\alpha}\frac{\partial c_\alpha}{\partial\boldsymbol{r}_w} f_T(z/z_\alpha), \tag{19}$$

(See [6] for the explicit form of f_L and f_T.) The asymptotic forms for large $z \gg (L^2\rho_0)^{1/3}$ are found to be

$$W_L \approx \frac{cZ_0\sqrt{\rho_0}}{2\sqrt{\pi z^3}}\sum_\alpha c_\alpha, \quad \boldsymbol{W}_\perp \approx \frac{cZ_0\sqrt{\rho_0}}{\sqrt{\pi z}}\sum_\alpha \frac{\partial c_\alpha}{\partial\boldsymbol{r}_w}. \tag{20}$$

I. The AC Conductivity at High Frequencies

As pointed out by Bane [3], the AC conductivity is no longer equal to the DC conductivity at very high frequencies and is approximately expressed by $\sigma/(1 - i\omega\tau)$, τ being the relaxation time of the metal. Our formulas in the frequency domain are still valid in such a case. The wake function, however, cannot be expressed by the two functions $f_L(\zeta)$ and $f_T(\zeta)$.

II. APPLICATIONS

Results of the application to various pipes are given in [6], such as the transient behavior of the wake, influence of the AC conductivity, dependence on the source/witness particle location for round, elliptic, rectangular, and hyperbolic pipes. Here, we shall show some of them and some more structures.

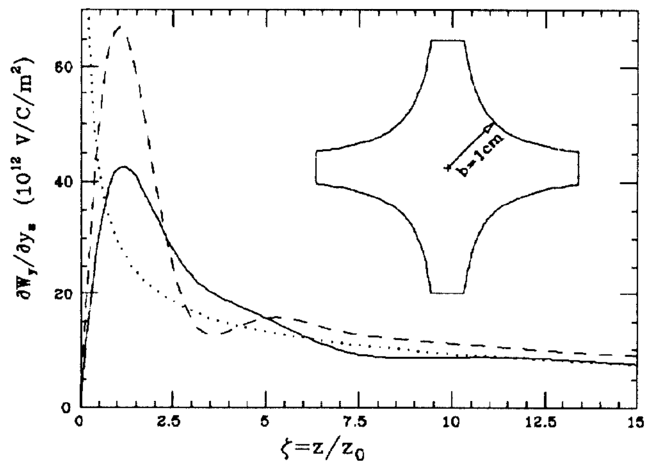

Figure 1: Transverse wake function (solid line) for a hyperbolic pipe with b=1cm. The dotted line is the asymptotic form and the dashed line is for the round pipe with radius 1cm.

Fig. 1 shows the transverse wake function for a hyperbolic pipe having a shape like the pole of a quadrupole magnet. The radius at the pole tip is b =1cm. (The area is cut at 2cm but the result is almost independent of the cut if it is larger than 1.5cm.) The horizontal axis is the normalized distance $\zeta = z/z_0$ with $z_0 = (b^2\rho_0)^{1/3}$ ($= 17\mu$m for copper). The solid line is $\partial W_y/\partial y_s$ for the hyperbolic pipe. The dotted line is the asymptotic form ($\propto 1/\sqrt{z}$). For comparison, the wake for the round pipe with radius 1cm is plotted in the dashed line . One finds the wake for the hyperbolic pipe is condiderably smaller than that for the round pipe in the short-range region but the difference is only slight in the asymptotic region (factor 0.835).

Next, let us discuss the dependence of the transverse asymptotic wake on the location of the witness particle $\boldsymbol{r}_w$ with $\boldsymbol{r}_s = 0$. The transverse wake is absent in the case of round pipes but this is not true in general.

The vertical asymptotic wake is plotted in Fig. 2 as a function of y_w for rectangular and elliptic pipes. The aspect ratio is indicated by the line modes as shown in the figure with crosses for the curves for rectangular pipes. The

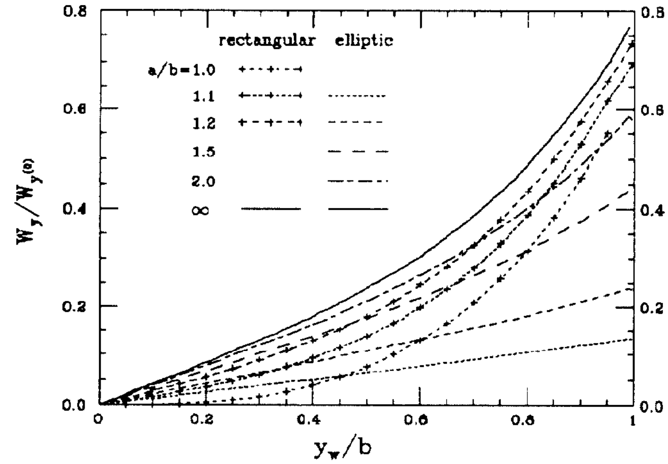

Figure 2: Vertical asymptotic wake vs. the location of the witness particle for rectangular and elliptic pipes of various aspect ratio with fixed vertical aperture.

vertical half aperture b is fixed in all cases. The wake W_y is normalized by $W_y^{(0)} = b[\partial W_y/\partial y_s]_{\text{round pipe},\boldsymbol{r}_s=0}$, which is the dipole wake for a round pipe when the source particle is near the pipe wall. From this figure we find the following facts. Firstly, W_y increases as y_w in rectangular pipes more rapidly than in elliptic pipes. Even for the square pipe a/b=1.0, W_y is almost the same as that for $a/b = \infty$, when the witness particle is close to the wall. Secondly, the y_w dependence is almost linear for elliptic pipes if $a/b \lesssim 1.5$ but W_y is still large near the wall unless a/b is very close to unity. For example, when a/b=1.2, W_y is about one quarter of the dipole wake of a round pipe with $y_s = b$.

These facts strongly suggest that the collimator for linear colliders has to be round.

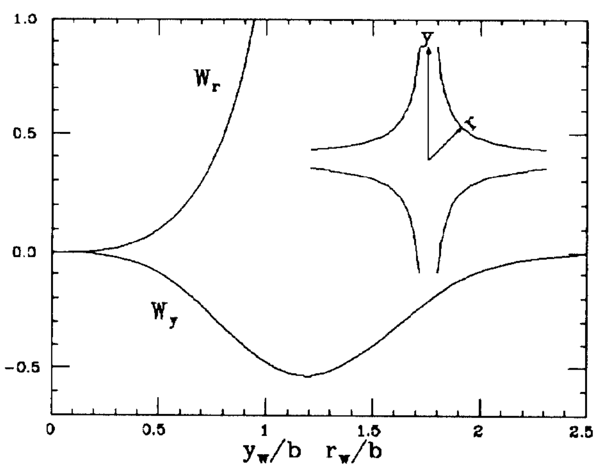

Figure 3: Transverse asymptotic wake for a hyperbolic pipe for $\boldsymbol{r}_w$ on the y-axis (W_y) and on the 45-degree line (W_r). The distance from the center to the pole is b.

Fig. 3 is a similar plot for hyperbolic pipe. Plotted is the transverse asymptotic wake for $\boldsymbol{r}_s = 0$ with the witness particle along the y-axis (W_y) and that along the 45-degree line (W_r). They are normalized by the same $W_y^{(0)}$ as in the previous plot. W_r becomes large near the pole face.

As we have seen in Fig. 1, the asymptotic wake for a hyperbolic pipe is nearly the same as that in a round pipe tangent at the pole face. This means that the wall current is concentrated on a part of the wall close to the beam. We have computed the asymptotic wake for a flat-face scraper with finite horizontal width $2a$ and the gap height $2b$. The result is shown in Fig. 4. Both the source and witness particle are at the center. The longitudinal wake W_L (long-dash) and four transverse wakes, $\partial W_x/\partial x_s$ (dot-dash), $\partial W_y/\partial y_s$ (dash), $\partial W_y/\partial y_w$ (dot), and $\partial W_y/\partial y_s + \partial W_y/\partial y_w$ (solid) are plotted against the width a. W_L is normalized by W_L of a round pipe of radius b, and the transverse wakes are normalized by $\partial W_y/\partial y_s$ for the round pipe. The limit $a \to \infty$ corresponds to the two parallel pipes. One finds that the wake is rather insensitive to the scraper width a and is even large when a is small.

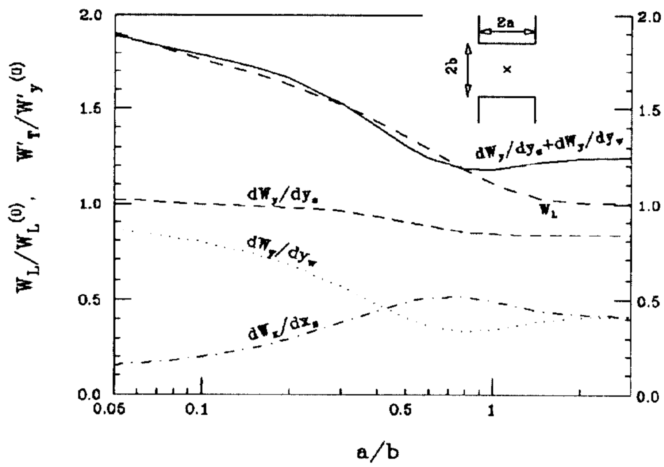

Figure 4: Asymptotic wake for finite width parallel collimator for $\boldsymbol{r}_w = \boldsymbol{r}_s = 0$. (width $2a$, gap height $2b$) Normalized by the corresponding wake for a round pipe with radius b.

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