An Analytical Treatment of Self Fields in a Relativistic Bunch of Charged Particles in a Circular Orbit

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Abstract

It is known that the electromagnetic field caused by a moving charge depends on its acceleration. Therefore, if a bunch of charged particles has a circular trajectory, the self fields in the bunch depend on the radius of curvature. We will treat these self fields analytically for a one-dimensional bunch, using the Liénard-Wiechert potentials. These depend on the retarded positions of the charges in the bunch. We will show that one only has to determine these positions explicitely for the endpoints of the bunch. The one-dimensional model predicts non-zero tangential and radial forces in the middle of the bunch which depend on its angular width and on its angular velocity. Expressions for these forces are presented. A comparison between the power loss due to coherent radiation and the tangential force exerted on the central electron of the bunch shows that there is a definite relation between these quantities.

I. INTRODUCTION

We consider a charge q in an arbitrary orbit. At time t', the charge is located at $\vec{r}'$, has velocity $\vec{\beta}$ and acceleration $\dot{\vec{\beta}}$. The electromagnetic field caused by this charge, experienced at time $t > t'$ and position $\vec{r}_1$ can be derived from the Liénard-Wiechert potentials [2] and reads

$$\vec{E}(\vec{r}_1,t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{\vec{n}-\vec{\beta}}{\gamma^2(1-\vec{\beta}\cdot\vec{n})^3\Lambda^2} + \frac{\vec{n}\times\{(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}\}}{c(1-\vec{\beta}\cdot\vec{n})^3\Lambda}\right]$$

$$\vec{B}(\vec{r}_1,t) = (\vec{n}/c)\times\vec{E}(\vec{r}_1,t),$$

with $\Lambda = c(t-t') = ||\vec{r}_1 - \vec{r}'||$ and $\vec{n} = (\vec{r}_1 - \vec{r}')/\Lambda$. The first term in the equation for $\vec{E}$ represents the usual Coulomb-like 'space charge field,' the second term the 'synchrotron radiation field' (containing the acceleration $\dot{\vec{\beta}}$ and being perpendicular to $\vec{n}$). The fact that the above equations relate the EM field at t to quantities at retarded time t' makes it difficult to express the total field at t for a bunch with arbitrary charge distribution in a general orbit. The retardation condition, which depends on the orbit path and observer position $\vec{r}_1$, must be solved to express the relation between $\vec{r}(t)$ and $\vec{r}'(t')$.

II. ONE-DIMENSIONAL BUNCH

The treatment presented in this paper is an overview of work reported in reference [1]. We look at the electromagnetic field for the specific case of a homogeneously charged 1D bunch in a circular orbit with radius R (see Fig. 1). The 'bunch angle' is denoted $\varphi_m = l/R$ with l the (longitudinal) size of the bunch. The (constant) rotation frequency is ω and the linear charge density is λ. We consider a reference charge e at an angular position φ_1 relative to the front side of the bunch, i.e. $-\varphi_m < \varphi_1 < 0$ (all angles will be taken positive in the direction of rotation). The force exerted on e is caused by all other charges in the bunch. One of those other charges is q, at angular position φ ($-\varphi_m < \varphi < 0$). At retarded time t', charge q emits a photon that reaches charge e at time t. Meanwhile, the bunch has rotated over an angle $-\theta_b = \omega(t-t')$, $\theta_b < 0$. The angular distance between q at t' and e at t is denoted $\theta = \theta_b + \varphi - \varphi_1$ (can be positive or negative). The retardation condition expresses the relation between θ_b and $(\varphi - \varphi_1)$. We obtain (for the case $\varphi_m < \pi - 2\beta$)

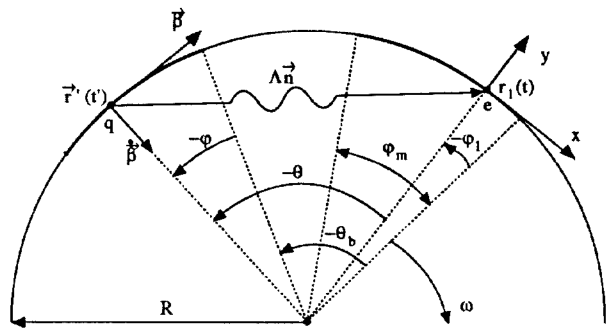

Figure 1: *1D bunch in a circular orbit.*

$$\varphi - \varphi_1 = |\theta_b| \pm 2\arcsin\left(\frac{|\theta_b|}{2\beta}\right), \quad -2\beta \le \theta_b < 0.$$

Note that charges both to the left and to the right of e contribute to the field, hence two values for $(\varphi - \varphi_1)$ exist for given θ_b.

We now consider an infinitesimal charge $dq = \lambda R d\varphi$ at angular position φ. It causes an electric field $d\vec{E}$ at the position of reference charge e and a force $d\vec{F}$ given by

$$d\vec{F} = e\{d\vec{E} + \beta\vec{e}_x \times (\vec{n}\times d\vec{E})\}.$$

where the force contribution by the magnetic field has also been taken into account (note that β is constant). Here, the coordinate system (x, y) has been used, with $\vec{e}_x$ the unit vector at $\vec{r}_1$ in the *tangential* direction, and $\vec{e}_y$ in the *radial* direction. The total force on e caused by the entire

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bunch is then found via integration

$$\vec{F} = \int_{bunch} d\vec{F} \equiv \lim_{\varepsilon \downarrow 0} \left(\int_{-\varphi_m}^{\varphi_1-\varepsilon} \frac{d\vec{F}}{d\varphi} d\varphi + \int_{\varphi_1+\varepsilon}^{0} \frac{d\vec{F}}{d\varphi} d\varphi \right).$$

In the limit $\varepsilon \downarrow 0$, the separate integrals are not finite, but only their sum is relevant. The x and y components of the force are given by

$$\mathcal{F}_x = \varphi_m \int_{bunch} \mathcal{E}_x d\varphi,$$
$$\mathcal{F}_y = \varphi_m \int_{bunch} \{\beta n_y \mathcal{E}_x + (1-\beta n_x)\mathcal{E}_y\} d\varphi,$$

where the dimensionless quantities $\mathcal{E}$ and $\mathcal{F}$ are defined according to

$$\mathcal{E} = \frac{4\pi\varepsilon_0 R^2}{q} E, \quad \mathcal{F} = \frac{4\pi\varepsilon_0 l}{e\lambda} F.$$

In order to find analytical expressions for $\mathcal{F}_x$ and $\mathcal{F}_y$, it would be convenient if $\mathcal{E}_x$, $\mathcal{E}_y$ and $\vec{n}$ could be expressed as functions of φ. However, these quantities are only known as a function of θ and are given by

$$n_x = -(\sin\theta)/\mathcal{W}, \quad n_y = (1-\cos\theta)/\mathcal{W},$$
$$\mathcal{W} = \sqrt{2-2\cos\theta},$$
$$\mathcal{E}_x = \frac{(2\beta^2-1)\sin\theta - 2\beta^2\sin(2\theta) + \mathcal{W}\beta(\beta^2-\cos\theta)}{(\beta\sin\theta+\mathcal{W})^3},$$
$$\mathcal{E}_y = \frac{(1+\beta^2\cos\theta)(1-\cos\theta) + \mathcal{W}\beta\sin\theta}{(\beta\sin\theta+\mathcal{W})^3}.$$

The retardation condition expresses θ as an implicit function of φ, with φ_1 and β as parameters. It turns out that it is impossible to express θ as a finite number of explicit functions in φ. As a solution to this problem, we simply choose θ_b as new integration variable. This is a very useful method, since both θ and φ are explicit functions of θ_b. The relation between θ and θ_b reads

$$\cos\theta = 1 - \frac{\theta_b^2}{2\beta^2}, \quad \sin\theta = \pm\frac{|\theta_b|}{\beta^2}\sqrt{\beta^2 - \tfrac{1}{4}\theta_b^2}.$$

As an example, the equation for the tangential force component becomes

$$\mathcal{F}_x = \varphi_m \lim_{\varepsilon \downarrow 0} \left(\int_{\theta_{b1}}^{\theta_{b2}} \mathcal{E}_x \frac{d\varphi}{d\theta_b} d\theta_b + \int_{\theta_{b3}}^{\theta_{b4}} \mathcal{E}_x \frac{d\varphi}{d\theta_b} d\theta_b \right).$$

By having changed the integration variable from φ (longitudinal position) to θ_b (representing time), the retardation condition now only has to be solved explicitely for the four endpoints of the integrals rather than for every single position within the bunch. We have for given β and φ_1

$$\theta_{b1} = \theta_b(\varphi = -\varphi_m), \quad \theta_{b2} = \theta_b(\varphi = \varphi_1 - \varepsilon),$$
$$\theta_{b3} = \theta_b(\varphi = \varphi_1 + \varepsilon), \quad \theta_{b4} = \theta_b(\varphi = 0).$$

III. TANGENTIAL FORCE

The expression for the tangential force $\mathcal{F}_x$ can now be found analytically. For this purpose, the variable v is introduced

$$v(\theta_b) = \left(1 - \frac{\theta_b^2}{4\beta^2}\right)^{-1/2},$$

and the tangential force reads (still assuming $\varphi_m < \pi - 2\beta$)

$$\mathcal{F}_x = \varphi_m\{\mathcal{P}_x(v_4, \beta) - \mathcal{P}_x(v_1, -\beta)\}, \quad v_i = v(\theta_{bi}),$$

with

$$\mathcal{P}_x(v,\beta) = \frac{1+\beta}{4}\sqrt{1-\frac{2}{v+1}} + \frac{1-\beta}{4}\sqrt{1+\frac{2}{v-1}} + \frac{\beta^2}{2}\sqrt{1-\frac{2\beta}{v+\beta}+\frac{\beta^2-1}{(v+\beta)^2}}.$$

Note that v_2 and v_3 do not appear in the expression for $\mathcal{F}_x$ because their contributions cancel in the limit $\varepsilon \downarrow 0$. This implies that the retardation condition only has to be solved for the two edges of the bunch.

In practice, we are mainly interested in forces near the centre of the bunch (denoted '0' for convenience). In case $\frac{1}{2}\varphi_m \ll 1-\beta$ (i.e. $\beta \ll 1$), we obtain for the tangential force in the middle of the bunch

$$\mathcal{F}_x(0) = -\tfrac{2}{3}\beta^3\gamma^4\varphi_m^2 + O(\varphi_m^4).$$

It is seen that the force is *unequal zero and negative*, i.e. points in a direction opposite to the bunch velocity. It can be shown that the minus sign is caused by a negative contribution originating from the synchrotron field. The space charge field gives a (three times smaller) positive contribution, which is also unequal zero as a result of the orbit curvature ($\varphi_m \neq 0$). In case $\gamma \gg 1$, the above approximation is not valid. Instead, the following expression must be used

$$\mathcal{F}_x(0) = -\left(\frac{4\varphi_m}{\sqrt{3}}\right)^{2/3} + \tfrac{1}{120}(12\varphi_m)^{4/3} + O(\varphi_m^2).$$

Again, the large negative term is caused solely by the synchrotron field. In this expansion we see that $\mathcal{F}_x(0)$ is mainly proportional to $R^{-2/3}$ and independent of γ. However, γ-dependency appears in higher order terms.

Both the above expressions for $\mathcal{F}_x(0)$ (based on expansions of v_1 and v_4) are in good agreement with numerical calculations, which solve v_1 and v_4 exactly.

IV. TANGENTIAL FORCE VS. POWER LOSS

The above results show that $\mathcal{F}_x(0) < 0$ over the full energy range $0 < \beta < 1$ and that the resulting bunch deceleration is caused entirely by the synchrotron field component. This leads to the thought that there could be a relation between the force $\mathcal{F}_x(0)$ and the power loss due to (synchrotron) radiation. The general relation between the power P_b lost by a bunch in circular motion and the average force $\langle F_x \rangle$ exerted on the particles in the bunch reads

$$P_b = -N\omega R \langle F_x \rangle,$$

with N the total number of particles. The power P_e radiated by a single charge e in circular motion is given by $P_e = \sum_{n=1}^{\infty} P_n$ with [3]

$$P_n = \frac{n\omega e^2}{4\pi\varepsilon_0 R}\left[2\beta^2 J'_{2n}(2n\beta) - (1-\beta^2)\int_0^{2n\beta} J_{2n}(x)dx\right],$$

and J_n the Bessel function of order n. The total power P_b radiated by a bunch with given charge distribution can be split into incoherent (P_{inc}) and coherent (P_{coh}) contributions. For the subsequent calculations, we assume $P_{inc} \ll P_{coh}$, which is valid for high current, low energy, bunched beams (e.g.: $\gamma = 10$, $\varphi_m = 0.2$ rad and $N = 2 \cdot 10^{10}$ gives $P_{inc}/P_{coh} \approx 2 \cdot 10^{-8}$). We then get for the scaled, average force representing the decelerating 'radiation reaction' caused by the coherent power loss of a homogeneously charged 1D bunch

$$\langle \mathcal{F}_x \rangle = \begin{cases} -\frac{2}{3}\beta^3\varphi_m^2 + O(\beta^3\varphi_m^4) & \text{for } \beta \ll 1, \\ -(3\varphi_m)^{2/3} + O(\varphi_m^{4/3}) & \text{for } \gamma \gg 1. \end{cases}$$

So, apart from a numerical factor close to 1, the average radiation reaction force $\langle \mathcal{F}_x \rangle$ is equal to the total tangential force $\mathcal{F}_x(0)$ exerted on the central electron in the bunch. In general, there is no *a priori* relationship between the average force and the force experienced by the central electron, but such a relation seems to exist in the present case.

V. RADIAL FORCE

The expression for the radial force $\mathcal{F}_y$ is found in a similar way as for the tangential force. We get

$$\mathcal{F}_y = \varphi_m[\mathcal{P}_y(v_1, -\beta) - \mathcal{P}_y(v_2, -\beta) - \mathcal{P}_y(v_3, \beta) + \mathcal{P}_y(v_4, \beta)]$$

$$\mathcal{P}_y(v, \beta) = \frac{1+\beta^2}{4}\ln\left(\frac{v-1}{v+1}\right) + \frac{\beta^2(1-\beta^2)}{2(v+\beta)},$$

and v_i defined as before. Contrary to the case of the tangential force, we cannot take the limit $\varepsilon \downarrow 0$ here since $\mathcal{F}_y$ is divergent. This is caused by the fact that the bunch has no radial dimension. As a solution, we think of the bunch as a sector (angle φ_m) of a 3D torus with major radius R (orbit radius) and minor radius a (bunch radius, $a \ll R$). Then, we must set [4]

$$\varepsilon \approx \frac{a}{2R}.$$

So, we calculate $\mathcal{F}_y$ according to the 1D model, but we use a finite value for ε that approximately takes the properties of a 3D bunch into account.

We now consider the value of $\mathcal{F}_y$ in the centre of the bunch. Assuming $\varepsilon \ll \varphi_m \ll 1$, expansions are used to find the most important contributions. We get

$$\mathcal{F}_y(0) = \begin{cases} (1+\beta^2)\varphi_m \ln(\varphi_m/2\varepsilon) + O(\varphi_m^3) & \text{for } \beta \ll 1, \\ \frac{4}{3}\varphi_m \ln(\varphi_m/2\varepsilon) + O(\varphi_m^{5/3}) & \text{for } \gamma \gg 1. \end{cases}$$

In the first case ($\beta \ll 1$), it turns out that the force is entirely due to the electric part of the space charge field. The magnetic force and the synchrotron field contribution can be neglected. Additionally, the expression for $\mathcal{F}_y(0)$ is in perfect agreement with results obtained from an EM-statics approach. In the second case ($\gamma \gg 1$), the force is mainly caused by the synchrotron field. In both cases, we see that $\mathcal{F}_y(0)$ is positive, i.e. points in a direction away from the centre of curvature. Moreover, $\mathcal{F}_y(0)$ is inversely proportional to R and almost independent of γ. Finally note that the above expressions for $\mathcal{F}_y(0)$ are in good agreement with numerical calculations, which solve v_1 through v_4 exactly.

VI. CONCLUSIONS

Self forces in a 1D bunch were calculated using Liénard-Wiechert field expressions. By choosing a convenient coordinate transformation, an analytical expression for the force vector has been found and it is shown that the retardation condition only needs to be solved explicitely for the two endpoints of the bunch. This can be done numerically or by making an analytical expansion in terms of the bunch angle. It follows that the self force in the middle of the bunch has non-zero radial and tangential components. For low energy bunches ($\beta \ll 1$), the tangential force is almost zero while the radial force has a finite value that is in perfect agreement with the result of EM statics. For high energy bunches ($\gamma \gg 1$), these forces reach a limiting value. Over the entire energy range, the tangential force points in a direction opposite to the bunch velocity and seems to be closely related to the coherent radiation reaction force.

VII. REFERENCES

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