

# On Coupling Impedances of Pumping Holes

Sergey S. Kurennoy

Superconducting Super Collider Laboratory\*  
2550 Beckleymeade Ave., Dallas, TX 75237 USA

## Abstract

Coupling impedances of a single small hole in vacuum-chamber walls have been calculated at low frequencies. To generalize these results for higher frequencies and/or larger holes one needs to solve coupled integral equations for the effective currents. These equations are solved for two specific hole shapes. The effects of many holes at high frequencies where the impedances are not additive are studied using a perturbation-theory method. The periodic versus random distributions of the pumping holes in the Superconducting Super Collider liner are compared.

## I. INTRODUCTION

Pumping holes and slots are very typical and numerous discontinuities of the vacuum chamber in accelerators. The contributions of a small hole to the beam-chamber coupling impedances at low frequencies (below the chamber cut-off) have been calculated analytically [1, 2], and the results coincide well with subsequent simulations and measurements, e.g. [3]. The approach is based on the Bethe theory of diffraction by small holes [4], which can be applied when the wavelength is large compared to a typical hole size, and the hole size is small compared to that of the beam-pipe cross section. Due to the impedance additivity below cut-off, this theory gives reasonable estimates for many holes at low frequencies.

In the present paper we attempt to generalize this approach for a single hole with dimensions comparable to or larger than those of the chamber cross section. This leads to integral equations that are solved for two particular cases.

At frequencies above cut-off the problem is more complicated since there is no additivity of contributions to the coupling impedance from different discontinuities. To study the impedance of many holes above cut-off we use a model based on the perturbation method. This model allows us to compare the impedance for periodic and random distributions of pumping holes in the Collider liner.

## II. INTEGRAL EQUATIONS

To calculate the coupling impedance we have to find the fields induced in the chamber by a given current perturbation, e.g., by a relativistic point charge. Taking as a zeroth

approximation the fields in the chamber without hole, one can consider them as incident electromagnetic waves on the hole. According to the Bethe theory [4], the fields diffracted by the hole can be obtained as those radiated by effective surface "magnetic" currents, which have to be introduced to satisfy the boundary conditions on the hole. Then integrating the fields along the beam path one can obtain coupling impedances. As a result, the longitudinal impedance of an arbitrary hole in the chamber with the circular cross section of radius  $b$  can be written as

$$Z(\omega) = -\frac{1}{2\pi bq} \iint_{hole} dS e^{-ikz} J_\varphi, \quad (1)$$

where  $k = \omega/c$  and  $J_\varphi$  is the Fourier-harmonic of the azimuthal component of the effective "magnetic" current induced by charge  $q$  in the hole. The impedance for an arbitrary chamber cross-section can be obtained using an expansion over eigenfunctions, e.g., [5], and for the longitudinal case it also includes only the effective-current component, which is transverse to the chamber axis. The effective surface current  $\vec{J}$  and charge  $\rho$  should satisfy integral equations

$$\begin{aligned} \frac{1}{2} E_r(\vec{r}) &= \frac{1}{4\pi} \iint_{hole} dS' (J_v \nabla_u G - J_u \nabla_v G), \\ \frac{1}{2} Z_0 H_\varphi(\vec{r}) &= \frac{1}{4\pi} \iint_{hole} dS' (c\rho \nabla_u G - ik J_u G), \end{aligned} \quad (2)$$

and the continuity condition  $\text{div} \vec{J} = i\omega\rho$ , where  $Z_0 = 120\pi\Omega$ ,  $G(R) = \exp(ikR)/R$ ,  $R = |\vec{r} - \vec{r}'|$ ;  $\vec{r}, \vec{r}' \in S$ ,  $\vec{J} = \vec{J}(\vec{r}')$ , and  $(u, v)$  are the local coordinates on the hole, with  $\vec{e}_v$  being parallel and  $\vec{e}_u$  transverse to the chamber axis  $z$ . According to [4],  $E_r$  and  $H_\varphi$  here are Fourier-harmonics of the beam fields on the wall in the chamber without hole, which are equal to  $E_r = Z_0 H_\varphi = Z_0 q \exp(ikz)/(2\pi b)$  in the round pipe.

In the case of a small hole ( $h \ll b$ ) at low frequencies ( $\omega \ll c/b$ ) one can consider the beam fields to be the same everywhere on the hole and reduce the problem to an electrostatic one [4]. Then the impedance can be obtained analytically in terms of hole polarizabilities [1, 2, 5]. In the case of a long slot (length  $l \gg b$ ) or when the transverse hole size is larger than the pipe radius, it does not work.

However, one can solve the general problem for two special cases. The first one is mostly of academic interest, namely, the infinitely long narrow slot, width  $w \ll b$ , in a perfectly conducting pipe. The only dependence on  $z$

\*Operated by the Universities Research Association, Inc., for the U.S. Department of Energy under Contract No. DE-AC35-89ER40486.

for such a slot is  $J_z, \rho \propto \exp(ikz)$ . Substituting  $J_z = c\rho$ ,  $J_\varphi = 0$ , one can reduce Eqs. (2) at frequencies  $\omega \ll c/b$  to a single equation for  $J_z$ , which has a solution

$$J_z(u, z) = \frac{Z_0 q}{\pi b} e^{ikz} \frac{u}{\sqrt{(w/2)^2 - u^2}},$$

where  $|u| < w/2$ . Since  $J_\varphi = 0$ , the impedance of such a slot vanishes, cf. Eq. (1). This answer follows as a limiting case from results for an elongated elliptic hole [1], and was also obtained in [6]. From the physical point of view, the charge drags the static field pattern along the chamber without producing any distortions. The field distortions could only be produced by the slot ends. It means that the low-frequency impedance of a long ( $l \gg b$ ) narrow slot is independent of slot length.

The second case is a narrow transverse gap of length  $g \ll b$  separating two pieces of the infinite beam pipe. Due to the axial symmetry there is no current dependence on  $\varphi$  when the beam goes along the axis. In the case of  $kg \ll 1$ , Eqs. (2) after integrating over  $\varphi$  are reduced to

$$\frac{1}{2} H_\varphi(z) = -ik \int_{-g/2}^{g/2} dz' e^{ikz'} K(z - z') J_\varphi(z'), \quad (3)$$

where  $K(2bx) = (1 + x^2)^{-3/2} {}_2F_1[1/2, 3/2, 1, (1 + x^2)^{-1}]$ . Eq. (3) has a solution

$$J_\varphi(x) \simeq \frac{4Z_0 q}{\pi k^2 g^2 \ln(16b/g)} \frac{x}{\sqrt{(g/2)^2 - x^2}},$$

which gives the low-frequency impedance of the gap

$$Z(\omega) = i \frac{Z_0 c}{2\omega b \ln(16b/g)}, \quad (4)$$

i.e., the capacitance  $C = 2\epsilon_0 b \ln(16b/g)$ . This result is natural since the gap cuts the image low-frequency currents in the wall and works as a capacitance. In a real accelerator chamber there are usually some electrical connections of chamber pieces separated by gaps, e.g., cavity walls or through the ground. Low-frequency currents flow through these connections, which have lower reactance in this frequency range than the gap.

A similar answer for  $C$  can be obtained from the plane electrostatic problem: find a capacitance per unit length of a gap  $g$  separating two half-planes. The problem can be easily solved by conformal mapping, and the result is  $2\epsilon_0 \ln(4\Lambda/g)/\pi$ . It includes the log-dependence on a cut-off parameter,  $\Lambda \gg g$ . Comparing to our cylindrical problem it seems natural to put  $\Lambda \simeq 2b$ , while the length is  $2\pi b$ . It gives us impedance (4) up to a factor of the order of 2.

To compare with numerical results we computed wakes in the chamber  $b = 2$  cm with a narrow gap  $g$  surrounded by a cavity with length  $l$  and depth  $h$ , by means of the code ABCI [7]. The cavity inductance is  $L = \mu_0 lh/(2\pi b)$ , and such a cavity-gap system will have resonances at  $\omega_r = 1/\sqrt{LC}$ , i.e., with wavelength  $\lambda_r = 2\sqrt{lh \ln(16b/g)}$ . Figure 1 shows a good agreement of this formula with ABCI results.

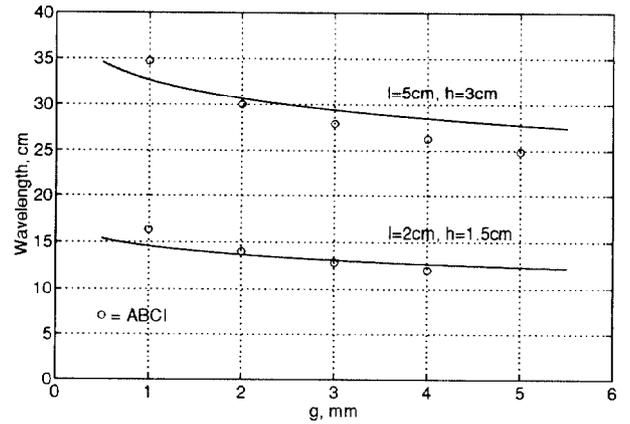


Figure 1. The resonance wavelength versus gap width.

### III. PERIODIC OR RANDOM STRUCTURE

The design of the liner inside the Superconducting Super Collider (SSC) Collider beam tube, which has to screen the cold chamber walls from the synchrotron radiation, anticipates a lot of small pumping holes. With the hole radius  $r = 1$  mm and  $b = 1.5$  cm the number of holes should be nearly 1500 per meter, and their total number in the ring is about  $10^8$ . There will be  $M = 15$  holes in one cross section, and such rows will be spaced by distance  $D = 1$  cm. The low-frequency impedances produced by these holes can be calculated as in Refs. [1, 2]:  $Z/n \simeq -i0.15 \Omega$  and  $Z_\perp \simeq -i18 \text{ M}\Omega/\text{m}$  for this specific case with the liner wall thickness 1 mm. To study effects of periodicity of liner holes at frequencies above cut-off we introduce a model that works for wavelengths large compared to the hole size, i.e., below  $f \sim c/(2h) \simeq 150$  GHz. Namely, we replace a row of  $M$  holes in one chamber cross section, which has a discrete axial symmetry, by an axisymmetric small enlargement with the triangular (in the longitudinal direction) cross section of depth  $h = r/2$  and base  $g = 2r$ . We will assume that the impedance of  $M$  holes in a row is that of such a discontinuity multiplied by azimuthal factor  $\phi = Mr/(\pi b)$ , cf. Ref. [1]. The model has small parameter  $\epsilon = h/(2b)$ , which is  $1/60$  in our case. So, one can apply the perturbation method for periodic structures of small discontinuities that was developed in [8], and its generalization to broken periodicity [9]. It makes use of an expansion over  $\epsilon$  in boundary conditions and gives the impedance at low frequencies and at resonances in an analytical form. The low-frequency impedance due to enlargements is

$$Z/n = -iZ_0 \epsilon^2 G/2 \sum_{p=1}^{\infty} p C_p^2 I_1(pG)/I_0(pG) + O(\epsilon^3), \quad (5)$$

where  $D$  is the structure period,  $G = 2\pi b/D$ ,  $I_m(x)$  are modified Bessel functions, and  $C_p$  are Fourier coefficients of the boundary shape,

$$C_p = (-1)^p 2g/D [\sin(\pi p g/2D)/(\pi p g/2D)]^2 \quad (6)$$

for triangular perturbations. The resonant frequencies  $f_{p,r}$  of the  $r$ -th radial mode, bandwidths  $(2\Delta f)_{p,r}$ , and impedances  $Z_{p,r}$  are

$$\begin{aligned} f_{p,r} &= c/(4\pi b) (pG + j_{0r}^2/pG) , \\ (2\Delta f)_{p,r} &= f_{p,r} \delta_{f_{p,r}} / (2b) \left[ 1 + (j_{0r}/pG)^2 \right] , \\ Z_{p,r}/n &= Z_0 \varepsilon^2 C_p^2 2b / \delta_{f_{p,r}} \left[ 1 + (j_{0r}/pG)^2 \right]^{-2} , \end{aligned} \quad (7)$$

where  $j_{0r}$  are roots of the Bessel function  $J_0(x)$  and  $\delta_f$  is the skin-depth at frequency  $f$ . From Eq. (5)  $(Z/n) \cdot \phi = -i0.17 \Omega$ , which is close to the exact answer for holes at low frequencies and justifies the choice of the model parameters. An important point is that Eqs. (5) and (7) work for an arbitrary distribution of perturbations on the ring; one has only to replace period  $D$  by the ring circumference  $2\pi R$  and take proper  $C_p$ . In this case  $N = 2\pi R/D$  identical perturbations are distributed along the ring. To take into account various insertions that violate the periodicity, e.g., interaction regions and warm pipe sections without holes, we consider the number of perturbations  $K \leq N$ . In general, new coefficients  $C_p^{(K)}$  are related to those  $C_p$  for a single perturbation on the ring. Let us assume that perturbations are randomly displaced from their positions in exactly periodic (period  $D = 2\pi R/N$ ) structure and the deviations have the Gauss distribution with dispersion  $(\delta D)^2$ . Then the averaged over distribution coefficients are

$$[C_p^{(K)}]^2 = F_p^{(K)}(\delta) C_p^2 , \quad (8)$$

$$F_p^{(K)}(\delta) = K(1 - \zeta_p) + \zeta_p F_p^{(K)} - \zeta_p (1 - \zeta_p) \frac{\sin 2\pi p K/N}{\sin 2\pi p/N}$$

where  $C_p$  are given by Eq. (6) with  $D \rightarrow 2\pi R$ ,  $\zeta_p = \exp[-(2\pi p \delta/N)^2]$  and  $F_p^{(K)} = [\sin(\pi p K/N)/\sin(\pi p/N)]^2$ . In the case of the exactly periodic structure,  $\delta = 0$ , we have  $\zeta_p = 1$  and  $F_p^{(K)}(0) = F_p^{(K)} \in [0, K^2]$ , with maxima  $K^2$  for  $p = Nl$ ,  $l = 0, 1, 2, \dots$ . If  $K = N$  in addition,  $F_p^{(K)} = N^2 \delta_{p, Nl}$ , i.e., all resonances  $p \neq Nl$  vanish, and the case of the structure with short period  $D = 2\pi R/N$  is recovered. Figure 2 shows the hole impedance for this case. The wall conductivity of copper at room temperature  $\sigma = 6 \cdot 10^7 (\Omega \text{ m})^{-1}$  was taken just for reference. Resonance values should be scaled  $\propto \sqrt{\sigma} \propto \sqrt{RRR}$ . Certainly, it gives the worst and, fortunately, unrealistic case, since for the real design the exact periodicity is broken.

Let us consider a more realistic model: blocks of length  $L = 15$  m containing the periodic hole structure (magnets with a regularly perforated liner inside) are separated by short insertions without holes (rf joints). Using the method of Ref. [9], one can calculate the damping of resonances in Figure 2 due to periodicity violation; it is by a factor 0.03–0.05. So, the maximal resonance values of  $\phi Z/n$  are 8–12  $\Omega$ , according to the block model.

One can go further and destroy periodicity inside blocks by placing holes not exactly in one transverse row, changing steps between rows, etc. The impedance estimate for this case (“random” hole distribution) can be obtained

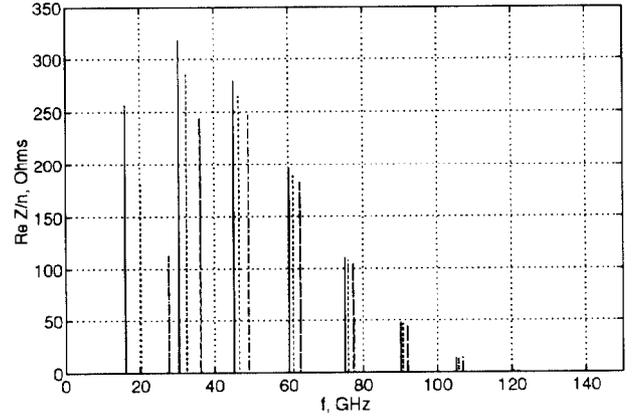


Figure 2. Resonances of the exactly periodic structure

from Eq. (8) with  $\zeta_p \rightarrow 0$ . Then  $F_p^{(K)}(\delta) \rightarrow K$ ; that means the incoherent sum of contribution from different perturbations, and due to overlapping of small resonances we get  $Re Z/n \simeq 0.2 \Omega$ , i.e., approximately the constant value in the frequency range above cut-off (7–50 GHz). In this case we have something like a broad-band impedance with  $Q = 1$ , since  $\phi Re Z/n(f > f_{cut}) \simeq \phi |Im Z/n|(f < f_{cut}) \simeq 0.2 \Omega$ .

## IV. CONCLUSIONS

The integral-equation approach allows one to calculate the impedance for two examples when the hole is not small. Unfortunately, these cases are of mostly academic interest.

The impedance of many small holes in the SSC Collider liner is studied at frequencies above cut-off using a model. It is shown that random hole distributions give lower impedance than periodic ones in this frequency range. It is reasonable to introduce some periodicity violation in the hole pattern.

## V. REFERENCES

- [1] S.S. Kurennoy, *Part. Acc.*, Vol. **39**, pp.1–13 (1992).
- [2] R.L. Gluckstern, Reports CERN SL/92-05.06(AP), Geneva (1992).
- [3] W. Chou and T. Barts, *Proceed. of this Conference*; E. Ruiz, et al., *ibid.*
- [4] H.A. Bethe, *Phys. Rev.*, Vol. **66**, pp.163–182 (1944).
- [5] S.S. Kurennoy, Report IHEP 92-84, Protvino (1992); *Proceed. of EPAC*, Berlin (1992).
- [6] S.A. Heifets, SLAC-AAS-Note-76, Stanford (1992).
- [7] Y.H. Chin, Report CERN SL/92-49 (AP), Geneva (1992).
- [8] S. Krinsky, *Proceed. of 11th Int. Conf. on Part. Accel.*, Geneva (1980), p.576.
- [9] S.S. Kurennoy, Report IHEP 88-10, Serpukhov (1988) and *Proceed. of EPAC*, Rome (1988), p.761.