Impedance Formalism for an Arbitrary Cumulative Instability

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Abstract

A formalism is developed for the analysis of collective instabilities in standing-wave systems. The analysis permits a unified treatment of the coupled-cavity free-electron laser, relativistic klystrons and other high power microwave sources. Coupling from both transverse and longitudinal beam motion is included in the calculation of the transverse and longitudinal impedances.

I. Introduction

An understanding of high-power microwave sources and their scalings is crucial to the future of high-energy electron-positron colliders. In fact, the tradeoffs between rf breakdown and beam break-up scalings [1] is responsible for the current consensus that future linear colliders should be powered by sources with an operating frequency in the 10-20 GHz range. Slow-wave devices are expected to produce the power levels required at the lower frequencies. However, they have also exhibited, at higher frequencies, what is, in fact, an intrinsic problem for such rf sources: when the structure is small enough to couple effectively to the longitudinal beam motion, it also couples effectively to the transverse motion. This results in, among other undesirable phenomena, beam break-up and pulse shortening [2].

To circumvent this scaling, the "coupling impedance" [3] of the desired longitudinal mode should scale independently from that for those TM modes which produce beam break-up. In effect, this requires circumventing the Panofsky-Wenzel theorem [4]. One method of accomplishing this has been proposed in the form of a "standing-wave" free-electron laser, in which transverse oscillatory motion is induced by a magnetic wiggler. Since the design orbit takes the beam off-axis, the premise of the Panofsky-Wenzel theorem fails, as does its conclusion.

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In this work we derive a formalism for quantifying such effects in an idealized cavity immersed in an arbitrary plane polarized magnetic field. The formalism in essence extends the wealth of work on longitudinal [6] and transverse [7] instabilities to include systems where the design particle orbit is curved within the cavity. In such "magnetized cavities" the coupling impedance describing longitudinal bunching can depend on the applied field. This permits one to enlarge the rf structure, so as to reduce undesirable transverse wakefields, while maintaining the desired longitudinal coupling. Previous workers [8] have calculated the coupling impedances in a cyclotron resonance maser with a traveling wave interaction region and a single cavity.

In the SWFEL, the power is produced in a series of uncoupled cavities (the rf is cutoff between the cavities), each of which is of order one wiggler oscillation in length. The FEL thus operates as a standing-wave device. The propagating beam provides the only coupling between the cavities. Numerical studies [9] of the SWFEL have examined phase sensitivity and longitudinal particle stability. In fact, the standing wave FEL has many similarities to the relativistic klystron, the main difference between them being that the FEL produces power through the coupling of the transverse wiggle oscillation with the transverse E field, while the klystron couples the longitudinal components (E_z with v_z).

II. Coupling Impedances

Assume that a bunch with unit charge enters a cavity at $t = 0$. The particles move transversely as well as longitudinally, due the presence of a magnetic field. Their trajectory and velocity inside the cavity is

$$\mathbf{r}(z) = x_1(z)\hat{\mathbf{x}} + z\hat{\mathbf{z}}, \mathbf{v}(z) = v_1(z)\hat{\mathbf{x}} + v_z\hat{\mathbf{z}}. \quad (1)$$

As the bunch moves through the cavity, it excites cavity modes. Without loss of generality, we may consider a single cavity mode. The vector potential of the mode can be represented as

$$\mathbf{A} = \frac{mc^2}{e} q(t)\mathbf{a}(\mathbf{r}), \quad (2)$$

where $\mathbf{a}(\mathbf{r})$ satisfies the appropriate boundary conditions on the wall and is normalized to the volume V of the cavity:

$$\int d^3r|\mathbf{a}(\mathbf{r})|^2 = V. \quad (3)$$

Also, $q(t)$ is the amplitude of the cavity excitation and can be determined from a knowledge of trajectory and velocity of the bunch:

$$q(t) = \frac{4\pi c}{V}\frac{e}{mc^2}\int_0^L dz' G(t - \frac{z'}{v_z})\mathbf{a}(x_1(z'), 0, z') \cdot \frac{\mathbf{v}_1(z')}{v_z}. \quad (4)$$

Here $G(t)$ is the cavity response Green function:

$$G(t) = \frac{1}{\Omega_\lambda}\sin\Omega_\lambda t e^{-\Omega_\lambda t/2Q_\lambda}\theta(t), \quad (5)$$

where $\Omega_\lambda^2 = \omega_\lambda^2 - (\frac{\omega_\lambda}{2Q_\lambda})^2$, with ω_λ the resonant frequency in the absence of damping and Q_λ the quality factor of the excited mode. $\theta(t)$ is the step function.

We assume a unit charge test particle enters the cavity at time $t = t_0$ with trajectory $\mathbf{r}_2(z)$ and velocity $\mathbf{v}_2(z)$. The test particle will experience the cavity mode excited by the first bunch. The longitudinal wakefield, defined to be the total energy loss of the test particle, is then

$$W^{\|}(t_0) = \frac{4\pi}{V}\int_0^L dz\int_0^L dz' G'(t_0 + \frac{z - z'}{v_z}) \mathbf{a}(x_1(z'), 0, z') \cdot \frac{\mathbf{v}_1(z')}{v_z}\mathbf{a}(x_2(z), 0, z) \cdot \frac{\mathbf{v}_2(z)}{v_z} \quad (6)$$

We are interested in the cases where the initial offsets for both the leading and trailing bunches are small. Then, the dominant contributions to the longitudinal wakefield can be computed assuming both bunches follow the same orbit:

$$W^{\|}(t_0) = \frac{4\pi}{V}|\int dz e^{i\Omega_\lambda z/v_z}\mathbf{a}(x(z), 0, z) \cdot \frac{\mathbf{v}(z)}{v_z}|^2 G'(t_0). \quad (7)$$

The longitudinal coupling impedance is

$$Z_\lambda^{\|}(\omega) = \int_{-\infty}^{+\infty} dt e^{i\omega t}W^{\|}(t) = R_\lambda^{\|}\frac{1}{1 + iQ_\lambda(\frac{\omega_\lambda}{\omega} - \frac{\omega}{\omega_\lambda})}. \quad (8)$$

In Eq. 8, the longitudinal shunt impedance $R_\lambda^{\|}$ is given by

$$\frac{R_\lambda^{\|}}{Q_\lambda} = \frac{4\pi}{\omega_\lambda V}|\int dz e^{i\Omega_\lambda z/v_z}\mathbf{a}(x(z), 0, z) \cdot \frac{\mathbf{v}(z)}{v_z}|^2. \quad (9)$$

This expression is valid for any particle orbit in the cavity, and includes both the transverse and longitudinal coupling.

As an example, consider the SWFEL in the limit that the betatron motion can be ignored (i.e., the betatron phase advance per cavity is small). The one-dimensional vector potential for a planar wiggler, $\mathbf{A}_w = \frac{mc^2}{e}a_w\cos k_w z\hat{\mathbf{x}}$, can be used to find the particle motion:

$$\mathbf{r}_\perp(z) = [x(0) - \frac{ca_w}{\gamma v_z k_w}\sin k_w z]\hat{\mathbf{x}} + O(\frac{a_w}{\gamma})^2,$$
$$\mathbf{v}(z) = -\frac{ca_w}{\gamma}\cos k_w z\hat{\mathbf{x}} + v_z\hat{\mathbf{z}} + O(\frac{a_w}{\gamma})^2. \quad (10)$$

Further calculations of the shunt impedance require a knowledge of the cavity mode. Here we take the modes of a rectangular cavity with transverse dimensions a and b, and longitudinal dimension d. $x(0) = a/2$ is at the center of the cavity. For the operational mode TE_{01p}, the shunt impedance is

$$\frac{R_{01p}^{\|}}{Q_{01p}} = \frac{4\pi}{\omega_{01p}V}(\frac{a_w d}{2\gamma\beta_z})^2. \quad (11)$$

For other modes (TE and TM), the shunt impedance is non-zero only if n is odd. With m even, and a TE mode,

$$\frac{R_{mnp}^{\|}}{Q_{mnp}} = 2\frac{R_{01p}^{\|}}{Q_{01p}}\frac{(\frac{n}{b})^2}{(\frac{m}{a})^2 + (\frac{n}{b})^2}(\frac{\sin\theta}{\theta})^2. \quad (12)$$

Here $\sin\theta/\theta$ is the largest transit time factor with $\theta = (\frac{\Omega_\lambda}{v_z} \pm \frac{p\pi}{d} \pm k_w)d/2$. When m is odd,

$$\frac{R_{mnp}^{\|}}{Q_{mnp}} = 2\frac{R_{01p}^{\|}}{Q_{01p}}\frac{(\frac{n}{b})^2}{(\frac{m}{a})^2 + (\frac{n}{b})^2}A_m^2(\frac{\sin\theta}{\theta})^2. \quad (13)$$

Here $A_m = (m\pi a_w/\gamma\beta_z k_w)a$, and $\theta = (\frac{\Omega_\lambda}{v_z} \pm \frac{p\pi}{d} \pm 2k_w)d/2$. When the mode is TM and m is even,

$$\frac{R_{mnp}^{\|}}{Q_{mnp}} = \frac{4\pi}{\omega_{mnp}V}\frac{2\delta^2 d^2}{\delta^2 + (\frac{p\pi}{d})^2}(1 \pm \frac{k_w}{2}\frac{p\pi}{d\delta^2})^2 A_m^2(\frac{\sin\theta}{\theta})^2. \quad (14)$$

Here, $\delta^2 = (\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2$ and the $\theta = (\frac{\Omega_\lambda}{v_z} \pm \frac{p\pi}{d} \pm k_w)d/2$. When m is odd,

$$\frac{R_{mnp}^{\|}}{Q_{mnp}} = \frac{4\pi}{\omega_{mnp}V}\frac{2\delta^2 d^2}{\delta^2 + (\frac{p\pi}{d})^2}(\frac{\sin\theta}{\theta})^2, \quad (15)$$

and $\theta = (\frac{\Omega_\lambda}{v_z} \pm \frac{p\pi}{d})d/2$.

The transverse impedance is found from the Fourier transform of the transverse kick per unit charge. For simplicity, we present results only for particle motion in x-direction (the wiggle plane). The transverse force experienced by a unit charge test particle is:

$$F_x = -\frac{1}{c}\frac{d\mathbf{A}_x}{dt} + \frac{\partial}{\partial x}(\frac{\mathbf{v}}{c} \cdot \mathbf{A}). \quad (16)$$

The net transverse kick is

$$K(t_0) = \int_{t_0}^{t_0 + L/v_z} dt F_x. \quad (17)$$

Assuming the front and end walls of the cavity are perpendicular to the axis, we can drop the surface term and find

$$K(t_0) = \frac{4\pi}{V}\int_0^L dz\int_0^L dz' G(t_0 + \frac{z - z'}{v_z}) \mathbf{a}(x_1(z'), 0, z') \cdot \frac{\mathbf{v}_1(z')}{v_z}\frac{\partial}{\partial x}\mathbf{a}(x_2(z), 0, z) \cdot \frac{\mathbf{v}_2(z)}{v_z} \quad (18)$$

Expanding $K(t)$ with respect to an initial offset x_0 of the leading bunch yields $W^{\perp}(t_0) = K(t_0)/x_0$:

$$W^{\perp}(t_0) = \frac{4\pi}{V}|\int_0^L dz e^{i\Omega_\lambda z/v_z} \frac{\partial}{\partial x}\mathbf{a}(x(z),0,z)\cdot\frac{\mathbf{v}(z)}{v_z}|^2 G(t_0). \tag{19}$$

Using Eqs. 5 and 19,

$$Z_\lambda^{\perp}(\omega) = -ic\int_{-\infty}^{+\infty} dt e^{i\omega t} W^{\perp}(t) = R_\lambda^{\perp}\frac{\omega_\lambda/\omega}{1 + iQ_\lambda(\frac{\omega_\lambda}{\omega} - \frac{\omega}{\omega_\lambda})}. \tag{20}$$

In Eq. 20, the transverse shunt impedance is given by

$$\frac{R_\lambda^{\perp}}{Q_\lambda} = \frac{4\pi c}{\omega_\lambda^2 V}|\int dz e^{i\Omega_\lambda z/v_z}\frac{\partial}{\partial x}\mathbf{a}(x(z),0,z)\cdot\frac{\mathbf{v}(z)}{v_z}|^2. \tag{21}$$

Comparing with the longitudinal shunt impedance, Eq. 9, we note a derivative of the vector potential with respect to the transverse position. Thus, we can read off the results for the transverse impedance. For the TE modes, when both n and m are odd,

$$\frac{R_{mnp}^{\perp}}{Q_{mnp}} = 2\frac{c}{\omega_{mnp}}(\frac{m\pi}{a})^2\frac{R_{01p}^{\|}}{Q_{01p}}\frac{(\frac{n}{b})^2}{(\frac{m}{a})^2 + (\frac{n}{b})^2}(\frac{\sin\theta}{\theta})^2. \tag{22}$$

When n is odd and m is even,

$$\frac{R_{mnp}^{\perp}}{Q_{mnp}} = 2\frac{c}{\omega_{mnp}}(\frac{m\pi}{a})^2\frac{R_{01p}^{\|}}{Q_{01p}}\frac{(\frac{n}{b})^2}{(\frac{m}{a})^2 + (\frac{n}{b})^2}A_m^2(\frac{\sin\theta}{\theta})^2. \tag{23}$$

For the TM modes, when both n and m are odd,

$$\frac{R_{mnp}^{\perp}}{Q_{mnp}} = (\frac{m\pi}{a})^2\frac{4\pi c}{\omega_{mnp}^2 V}\frac{2\delta^2 d^2}{\delta^2 + (\frac{p\pi}{d})^2}(1 \pm \frac{k_w}{2}\frac{p\pi}{d\delta^2})^2 A_m^2(\frac{\sin\theta}{\theta})^2. \tag{24}$$

When n is odd and m is even, we get the well-known transverse BBU impedance:

$$\frac{R_{mnp}^{\perp}}{Q_{mnp}} = (\frac{m\pi}{a})^2\frac{4\pi c}{\omega_{mnp}^2 V}\frac{2\delta^2 d^2}{\delta^2 + (\frac{p\pi}{d})^2}(\frac{\sin\theta}{\theta})^2, \tag{25}$$

For TM modes there is off-resonant DC deflection, which will be examined in a future paper.

Using the above definitions for the impedances, BBU calculations proceed in the usual manner. For example, in the long beam, high Q limit, both transverse ($y = x$) or longitudinal ($y = \tau$, the delay in arrival time with respect to the synchronous electron) BBU from a single cavity mode can be found from the coupled equations:

$$(\frac{\partial}{\partial t} + v_z\frac{\partial}{\partial z})^2 y = a(z,t), \tag{26}$$

$$\frac{\partial^2 a}{\partial t^2} + \frac{\omega_\lambda}{Q_\lambda}\frac{\partial a}{\partial t} + \omega_\lambda^2 a = Cy(z,t), \tag{27}$$

where, with a beam current I and a single cavity transit time T_0, the constant C is given by

$$C = \begin{cases} \frac{eI}{\gamma^3 m v_z^2 T_0}\omega_\lambda^3\frac{R_\lambda^{\|}}{Q_\lambda^{\|}} & \text{longitudinal,} \\ \frac{eI}{\gamma m c T_0}\omega_\lambda^2\frac{R_\lambda^{\perp}}{Q_\lambda^{\perp}} & \text{transverse.} \end{cases} \tag{28}$$

III. Conclusions

In a typical SWFEL design, the interaction cavity is highly overmoded. As a consequence, a realistic wake consists of a number of superimposed modes. These modes will not be given by the idealized cavity modes described herein, but rather need to be calculated numerically for the particular structure. These more realistic modes will produce some BBU growth through the conventional mechanism (that a particle slightly off axis couples to an E_z field), and will also generate BBU through the magnetized cavity mechanism described herein. An analysis of BBU which includes both realistic cavity modes and wiggling particle trajectories is a topic for future research.

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