

# A Generalized Model for Parametric Coupling of Longitudinal Modes in Synchrotrons

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## Abstract

Observations of nonlinear coupling of longitudinal modes in the Fermilab TEVATRON have been recently identified as a manifestation of parametric coupling.<sup>1</sup> In this model, a spatially uniform, finite-amplitude, longitudinal perturbation is applied to the beam, which then decays into two or more daughter waves as a result of this coupling. Selection rules are derived which are based on frequency and phase matching conditions. These selection criteria are obtained using a multiple time-scale expansion of the weakly non-linear Vlasov equation governing longitudinal motion.

## INTRODUCTION

Parametric decay is a wave-wave scattering process whereby a driving wave applied to a system can transfer energy into oscillatory modes of the system. For our case, the driving wave is an external voltage applied to a coasting, unbunched beam. The oscillatory modes are the longitudinal modes of the beam. Only those modes which obey the selection rules of parametric decay are allowed to be excited. This phenomena was observed in the Fermilab Tevatron while doing a longitudinal beam transfer function measurement of 150 Gev beam.

## EXPERIMENTAL OBSERVATION

The setup for the beam transfer function measurement was to use a network analyzer to drive the beam via a longitudinal kicker, while monitoring the response with a wideband resistive wall pickup. The return signal from the pickup was passed through a splitter so that the beam spectrum could be observed concurrently on a spectrum analyzer.<sup>2</sup> During this study the proton beam intensity was  $5 \times 10^{12}$  and  $\sigma_p/p$  was about  $2 \times 10^{-3}$ . The network analyzer scans were done at harmonics of the revolution frequency (47.7 kHz) with a span of 500 Hz. The spectrum shown in Figure 1. is the beam response to a scan with a center frequency of 47.7 MHz. Figure 1a. shows the high frequency end of the spectrum and Figure 1b. shows the low frequency end. There were no excited harmonics in the middle frequency range from 2-45 MHz.

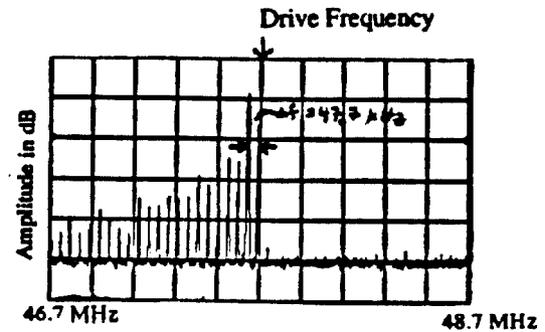


Figure 1a. High Frequency Spectrum

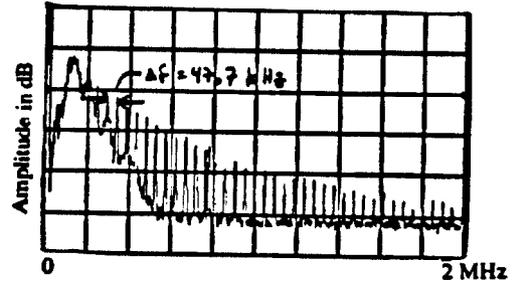


Figure 1b. Low Frequency Spectrum

The observed spectrum shows some remarkable features. The beam has responded to the driving voltage at multiple frequencies, and all of this harmonic content is at frequencies lower than that of the drive. There also appears to be a mirror symmetry between the high and low frequency ends of the spectrum (disregarding the noise at very low frequencies). These properties indicate a nonlinear mechanism, and are a natural consequence of parametric coupling. The overall structure suggests the requirement that energy be transferred from the driving wave to pairs of longitudinal modes whose individual frequencies sum up to the that of the pump wave,  $\omega_{m1} + \omega_{m2} = \omega_{drive}$ .

## THEORY

The system is described with the Vlasov equation written in the conjugate variables of longitudinal motion:

$$\frac{\partial f}{\partial t} + \dot{\theta} \frac{\partial f}{\partial \theta} + \dot{\epsilon} \frac{\partial f}{\partial \epsilon} = 0$$

where  $\dot{\theta} = \omega_0 + k_0 \epsilon$  is the revolution frequency, and  $\dot{\epsilon}$  is due to wakefield effects. Perturbing the ideal beam distribution

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function, the perturbative term can be expressed as an expansion in longitudinal modes:

$$f = f_0 + \sum_m f_m e^{im\theta}$$

The wakefield potential may also be expanded in longitudinal modes:

$$\dot{\mathcal{E}} = \frac{e\omega_0}{2\pi} \sum_m U_m e^{im\theta}$$

The phase matching condition comes from writing the equation for one Fourier component:

$$\frac{\partial f_m}{\partial t} + im\dot{\theta} f_m + \frac{e\omega_0}{2\pi} \frac{\partial f_0}{\partial \mathcal{E}} U_m + \frac{e\omega_0}{2\pi} \sum_{n+k=m} U_n \frac{\partial f_k}{\partial \mathcal{E}} = 0$$

This condition is  $n+k=m$ , the sum of the phases of the daughter waves must equal the phase of the driving wave.

The system can be naturally broken down into different time scales. The normal modes oscillate relatively rapidly compared to the slower time scale for the growth or decay of power in these modes. Thus, a multiple time scale perturbation expansion can be done, treating the time scales as independent variables.

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau_0} + \lambda \frac{\partial}{\partial \tau_1} + \dots$$

$$f_m = \lambda f_m^1 + \lambda^2 f_m^2 + \dots \quad \text{where } f_m^1 = \hat{f}_m(\tau_1) e^{-i\omega_m \tau_0}$$

$$U_m = \lambda U_m^1 + \lambda^2 U_m^2 + \dots \quad \text{where } U_m^1 = \hat{U}_m(\tau_1) e^{-i\omega_m \tau_0}$$

Making these substitutions, the first order equation is the standard linear dispersion relation<sup>3</sup>:

$$1 = \frac{(e\omega_0)^2}{2\pi i} Z \int \frac{\frac{\partial f_0}{\partial \mathcal{E}}}{\omega_m - m(\omega_0 + k_0 \mathcal{E})} d\mathcal{E}$$

The second order equation yields a dispersion relation describing the parametric coupling resonance, as well as the frequency matching condition required for coupling to occur. It also gives an expression for the growth rate of power in a given mode<sup>4</sup>. The complete second order equation contains both resonant and non-resonant terms. The multiple time scale perturbation expansion allows removal of secular behavior from the system by setting the sum of the resonant terms to zero. Identification of resonant terms gives us the frequency matching condition,  $\omega_n + \omega_k = \omega_m$ . If the frequency of the driving wave is  $\Omega_0$ , then  $\omega_n + \Omega_0 = \omega_m$  must be obeyed in order to have energy transferred into a pair

of longitudinal modes. The secular equation may be solved for the growth rate of the amplitude of oscillation of mode  $m$ :

$$\frac{\partial \hat{I}_m(\tau_1)}{\partial \tau_1} = \frac{-e\omega_0}{2\pi} \hat{I}_n V_0 \times \int \frac{d\mathcal{E}}{\omega_n + \Omega_0 - m\gamma} \frac{\partial}{\partial \mathcal{E}} \left( \frac{\frac{\partial f_0}{\partial \mathcal{E}}}{\Omega_0 - k\gamma} \right) \Bigg/ \int \frac{\frac{\partial f_0}{\partial \mathcal{E}} d\mathcal{E}}{(\omega_m - m\gamma)^2}$$

where  $V_0$  is the amplitude of the driving voltage and

$\gamma = \omega_0 + k_0 \mathcal{E}$ . Note that the growth rate is proportional to  $V_0$ .

The dispersion relation near the resonance can be found if the equation for the growth rate is also written for mode  $n$ . Now there are two coupled equations which may be solved using

normal mode analysis. Let  $\hat{I}_m(\tau_1) = A e^{\nu \tau_1}$  and

$\hat{I}_n(\tau_1) = B e^{\nu \tau_1}$ , ( $\nu$  being the frequency offset from resonance) then:

$$\nu^2 \int \frac{\frac{\partial f_0}{\partial \mathcal{E}} d\mathcal{E}}{(\omega_m - m\gamma)^2} \int \frac{\frac{\partial f_0}{\partial \mathcal{E}} d\mathcal{E}}{(\omega_n - n\gamma)^2} = \left( \frac{e\omega_0}{2\pi} \right)^2 mn(k_0 V_0)^2 \times \int \frac{\frac{\partial f_0}{\partial \mathcal{E}} d\mathcal{E}}{(\omega_m - m\gamma)(\omega_n - n\gamma)^2} \int \frac{\frac{\partial f_0}{\partial \mathcal{E}} d\mathcal{E}}{(\omega_n - n\gamma)(\omega_m - m\gamma)^2}$$

It is possible to get this same result using a different method. The complete dispersion relation can be found by taking the Fourier transform of the Vlasov equation for one Fourier component, substituting a driving voltage for one of the wakefield terms, and then making substitutions to cancel the current terms on both sides of the equation<sup>5</sup>:

$$\left[ 1 - Z \frac{(e\omega_0)^2}{2\pi i} \int \frac{\frac{\partial f_0}{\partial \mathcal{E}} d\mathcal{E}}{\omega_m - m\gamma} \right] \left[ 1 - Z \frac{(e\omega_0)^2}{2\pi i} \int \frac{\frac{\partial f_0}{\partial \mathcal{E}} d\mathcal{E}}{\omega_n - n\gamma} \right] = - \frac{(e\omega_0)^3}{(2\pi)^2} nk_0 V_0 Z \int \frac{\frac{\partial f_0}{\partial \mathcal{E}} d\mathcal{E}}{(\omega_m - m\gamma)(\omega_n - n\gamma)^2} \times \frac{(e\omega_0)^3}{(2\pi)^2} mk_0 V_0 Z \int \frac{\frac{\partial f_0}{\partial \mathcal{E}} d\mathcal{E}}{(\omega_m - m\gamma)^2(\omega_n - n\gamma)}$$

If this complete dispersion relation is expanded about the resonant frequency, the final expression is the same as that found using the time perturbation technique.

## CONCLUSION

The theory of parametric coupling has been adapted to the accelerator context in order to present a possible explanation for experimental observations. The selection rules which come out of the analysis are consistent with the characteristics of the beam spectra. Pertinent features of the physical system such as the growth rate and the dispersion relation have been calculated. A program is being developed to explore the behavior of the dispersion relation.

## REFERENCES

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