# Bunch Lengthening effect and Localized Impedance

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*Abstract*

The effect of the localized longitudinal impedance is investigated by using the Vlasov equation approach. The motivation is trying to explain the observed discrepancy between the analytical study and numerical simulation on the bunch lengthening phenomenon. The $\nu_s = m/2l$ resonances are found to play a noticeable role. As another application of this treatment, synchro-betatron resonances are also recovered.

## I. INTRODUCTION

The discrepancy of the analytical theories (the perturbation techniques, in particular) and the computer simulation on transverse coherent instability has been noticed and was successfully explained by considering the coherent synchro-betatron resonances due to the localized impedance[1][2][3]. When the combination of betatron tune $\nu_\beta$ and synchrotron tune $\nu_s$ is close to integer or half-integer, *i.e.* $\nu_\beta \pm l\nu_s \approx m/2$, the predictions on the mode coupling by two approaches are different. The reason behind this is that the regular Vlasov technique treats the impedance as a distributed one, the wake "force" acts on beam all along the orbit, but a typical computer simulation treats the impedance as a localized one, the wake force just kicks the beam once a turn at the particular location.

The same arguments may be applied to the longitudinal case in the study of the bunch lengthening phenomenon. One possibility to explain the discrepancy between the mode analysis and tracking is tracking uses localized impedance. In this paper, we investigate the longitudinal coherent resonance effect due to the longitudinal localized impedance by using a matrix technique[4]. For one simple particle distribution mode, the water-bag model, the resonance stopbands widths are calculated. The results are what one would expect from the pure analysis side. The further studies on the tracking side is necessary, but are not included in this paper. Also, we extend our studies to the transverse case where the synchro-betatron resonances are re-obtained by our approach.

## II. THE VLASOV TECHNIQUE

We consider a single bunch particle beam described by the Vlasov equation[5]

$$\frac{\partial\psi}{\partial s} + \frac{\omega_s}{c}\frac{\partial\psi}{\partial\phi} - \frac{e}{EC}V(z,s)\frac{\partial\psi}{\partial\delta} = 0, \tag{1}$$

where $\psi(r,\phi;s)$ is beam distribution in phase space which consists of a pair of dynamical variables $z$ and $\delta$. The action-angle variables $(r,\phi)$ are related to $(z,\delta)$ through $z = r\cos\phi$, $\frac{\eta c}{\omega_s}\delta = r\sin\phi$; $C$ is the circumference of the ring, $E$ is the beam energy, $\omega_s$ is the synchrotron frequency and $\eta$ is the slippage factor.

We assume the particle beam experiences a localized "kick" due to the interactions between the beam and the surroundings at the location $s = 0$, where $s$ is the distance along the orbit. The particle energy loss in one turn is (here the short range wake field is assumed)

$$V(z,s) = \frac{eC}{2\pi}\delta(s)\int_{-\infty}^{\infty} d\omega\tilde{\rho}(\omega,s)e^{\frac{i\omega z}{c}}Z_0^{\parallel}(\omega), \tag{2}$$

where $\tilde{\rho}(\omega,s)$ is the Fourier transform of $\rho(z)$, the beam distribution in $z$-axis.

Consider a beam with an unperturbed distribution $\psi_0$ which is executing collective oscillation due to the interaction of the wake fields. Let the collective oscillation be described by a small distribution perturbation $\psi_1(r,\phi;s)$, i.e.

$$\psi(r,\phi;s) = \psi_0(r) + \psi_1(r,\phi;s). \tag{3}$$

Keeping only the first order terms in $\psi_1$, we get the linearized Vlasov equation

$$\frac{\partial\psi_1}{\partial s} + \frac{\omega_s}{c}\frac{\partial\psi_1}{\partial\phi} - \frac{e^2C}{2\pi E}\delta(s)\sin\phi\frac{d\psi_0}{dr}$$
$$\times\int_{-\infty}^{\infty} d\omega Z_0^{\parallel}(\omega)e^{i\frac{\omega r\cos\phi}{c}}\int_0^{\infty} dr'r'\int_0^{2\pi} d\phi' e^{-i\frac{\omega r'\cos\phi'}{c}}\psi_1 = 0, \tag{4}$$

Here the potential well distortion effect is not included, the study of which is not the goal of our study.

We consider a simple model of $\psi_0$, the water-bag model

$$\psi_0(r) = \frac{4N\eta c}{\pi\omega_s\hat{z}^2}H(\frac{\hat{z}}{2} - r), \tag{5}$$

where $H$ is the Heaviside step function. The advantage of using the water-bag model is avoiding introducing the radial mode of collective motion, but does not severely limit the generality of the studies.

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Observing the fact that the perturbations occur only at the edge of the bag, we can write the perturbation as

$$\psi_1(r,\phi;s) = \delta(r-\frac{\hat{z}}{2}) \sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \alpha_l(s) e^{il\phi}. \qquad (6)$$

Substituting Eq.(6) into Eq.(4) gives an infinite set of equations describing the coupled motion for all $l$. We single out the $l$-th component

$$\frac{d\alpha_l(s)}{ds} + il\frac{\omega_s}{c}\alpha_l(s) - \delta(s) \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} K_{lk}\alpha_k(s) = 0, \qquad (7)$$

where the matrix $K$ has been defined by

$$K_{lk} = l\frac{2N\eta e^2 cC}{\pi^2 \nu_s E\hat{z}^2} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} i^{l-k} \int_{-\infty}^{\infty} d\omega \frac{Z_0^{\|}(\omega)}{\omega} J_l(\frac{\omega\hat{z}}{2c}) J_k(\frac{\omega\hat{z}}{2c}) \qquad (8)$$

which has the following properties:

$$K^2 = 0,$$

$$K_{-l,k} = -K_{lk}. \qquad (9)$$

We define an infinite dimension vector, whose $l$-th component is

$$\tilde{X}_l(s) = \alpha_l(s). \qquad (10)$$

Then the Eq.(7) becomes

$$\frac{dX_l(s)}{ds} + \sum_k R_{lk}X_k(s) + \delta(s)\sum_k K_{lk}X_k(s) = 0, \qquad (11)$$

where

$$R_{lk} = i\frac{2\pi l\nu_s}{C}\delta_{lk} \qquad (12)$$

The action of the impedance is obtained by integrating Eq.(11) through $s=0$, so the map from $s=0^-$ to $s=0^+$ is

$$X_l(0^+) - X_l(0^-) = -\sum_k K_{lk}X(0^-). \qquad (13)$$

Between the location of the impedance, the different $\alpha_l$'s are decoupled, the map from $s=0^+$ to $s=C^-$ is

$$X_l(C^-) = \sum_k S_{lk}X(0^+), \qquad (14)$$

where

$$S_{lk} = e^{-i2\pi l\nu_s}\delta_{lk}. \qquad (15)$$

Therefore the total map for one turn is

$$T = S(I-K), \qquad (16)$$

Where $I$ is the indentity matrix. The diagonal matrix $R$ describes the action between the location of impedance, and $K$ describes the localized "kick".

In the absence of the impedance, the total map $T=R$ whose eigenvalues are

$$\lambda_l = S_{ll} = e^{-i2\pi l\nu_s}. \qquad (17)$$

These correspond to the eigen modes of the unperturbed motion and the beam are always stable.

In the presence of the impedance (wake filed), as the beam current is increased, the eigenvalues of the total matrix $T$ are more and more perturbed. If one of them acquires an absolute value larger than one, the beam motion becomes unstable.

Without considering the coupling among the different modes, we keep only the $l$-th and the ($-l$)-th elements in the transformation matrix. The reason that it is necessary to keep ($-l$)-th terms is we must observe the property $K^2=0$ which guarantees we can use either $X(0^-)$ or $X(0^+)$ on the right hand side of Eq.(14).

The simplified $2\times 2$ matrix becomes

$$T = (-1)^m \begin{pmatrix} e^{-i2\pi l\Delta}(1+i\epsilon_l) & ie^{-i2\pi l\Delta}\epsilon_l \\ -ie^{-i2\pi l\Delta}\epsilon_l & e^{-i2\pi l\Delta}(1-i\epsilon_l) \end{pmatrix}, \qquad (18)$$

where

$$\Delta = \nu_s - \frac{m}{2l}, \qquad (19)$$

and

$$\epsilon_l = l\frac{2N\eta e^2 cC}{\pi^2\nu_s E\hat{z}^2}\int_{-\infty}^{\infty} d\omega \frac{ImZ_0^{\|}(\omega)}{\omega} J_l^2(\frac{\omega\hat{z}}{2c}), \qquad (20)$$

which is a real number.

The eigenvalues of the simplified matrix are then determined by the secular equation

$$\lambda^2 - 2(-1)^m[\cos(2\pi l\Delta) + \epsilon_l \sin(2\pi l\Delta)]\lambda + 1 = 0. \qquad (21)$$

One of the eigen values has absolute value larger than one and therefore the coherent instability occurs if

$$|\cos(2\pi l\Delta) + \epsilon_l\sin(2\pi l\Delta)| > 1. \qquad (22)$$

The stopband width at the region near the resonance $\nu_s = m/2l$ is approximately

$$\delta\nu_l = \frac{|\epsilon_l|}{\pi l} = \frac{2N\eta e^2 cC}{\pi^3\nu_s E\hat{z}^2}\left|\int_{-\infty}^{\infty} d\omega \frac{ImZ_0^{\|}(\omega)}{\omega} J_l^2(\frac{\omega\hat{z}}{2c})\right|, \qquad (23)$$

For the broad band impedance (valid for a diffraction impedance model)

$$Z_0^{\|}(\omega) = R_0\left|\frac{\omega_0}{\omega}\right|^{1/2}[1+sgn(\omega)i], \qquad (24)$$

the stopband width is

$$\delta\nu_l = \frac{8\sqrt{2}\Gamma(l-\frac{1}{4})}{\pi^2\Gamma^2(\frac{1}{4})\Gamma(l+\frac{5}{4})}\frac{\Upsilon}{\nu_s}, \qquad (25)$$

where

$$\Upsilon = \frac{\eta e I R_0}{E}(\frac{C}{\hat{z}})^{3/2}. \qquad (26)$$

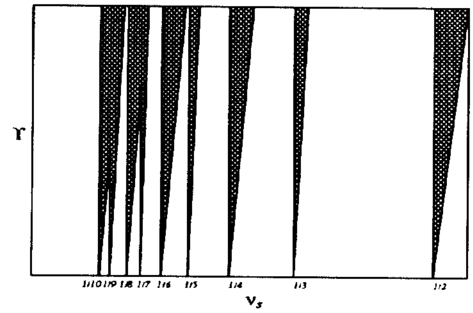


FIG. 1. The longitudinal stopbands. The shaded regions are unstable. Resonance structures are displayed by this diagram.

For SLAC SPEAR ring, $E = 3$GeV, $\nu_s = 0.042$, $\hat{z} = 2.68$cm and $I = 20$mA, $\delta\nu_6 \approx 0.005$.

## III. TRANSVERSE CASE AND SYNCHRO-BETATRON RESONANCES

We can also apply the same technique to the transverse case, which has been already extensively studied[1][2][3]. Here we just give some main results. For the transverse motion the Vlasov equation is:

$$\frac{\partial\psi}{\partial s} + \frac{\omega_\beta}{c}\frac{\partial\psi}{\partial\theta} - \frac{e}{EC}V_y(z,s)\frac{\partial\psi}{\partial p_y} + \frac{\omega_s}{c}\frac{\partial\psi}{\partial\phi} = 0, \qquad (27)$$

where the transverse dynamical variables, both the regular and the action-angle, are $y = r_y\cos\theta$, $-\frac{cp_y}{\omega_\beta} = r_y\sin\theta$, where $\omega_\beta$ is the betatron frequency. The localized transverse "kick" is defined by

$$V_y(z,s) = -i\frac{eC}{2\pi}\delta(s)\int_{-\infty}^{\infty} d\omega\tilde{\rho}(\omega,s)e^{\frac{i\omega z}{c}}Z_1^\perp(\omega), \qquad (28)$$

where the Fourier component $\tilde{\rho}(\omega)$ only comes from the contribution of the longitudinal distribution.

The distribution function is assumed as:

$$\psi(r,\phi;r_y,\theta;s) = f(r_y)\frac{N\eta c}{2\pi\omega_s\hat{z}}\delta(r-\frac{\hat{z}}{2})$$

$$-D\frac{df(r_y)}{dr_y}\cos\theta\delta(r-\frac{\hat{z}}{2})\sum_{p=1,-1}\sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty}\alpha_l^p(s)e^{il\phi}. \qquad (29)$$

where $\hat{z}$ is bunch length.

The first term of right hand side of Eq.(29) is the unperturbed distribution; $f(r_y)$ is its transverse part and the air-bag model is assumed for its longitudinal part. For the perturbed part the index $p$ takes only values 1 and -1. This because only the dipole motion is assumed for the transverse perturbation which is $-D\frac{df(r_y)}{dr_y}\cos\theta$, where $D$ is the dipole displacement of the distribution.

Following the procedures described in the last section, we defined a vector

$$\tilde{Y}_l^p(s) = \alpha_l^p(s), \qquad (30)$$

for which the one turn map is

$$T = S(I-K), \qquad (31)$$

where

$$S_{lk}^p = e^{-i2\pi(p\nu_\beta+l\nu_s)}\delta_{lk}, \qquad (32)$$

and

$$K_{lk}^d = p\frac{Ne^2cC}{8\pi^2\nu_\beta E}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} i^{l-k}\int_{-\infty}^{\infty} d\omega Z_1^\perp(\omega)J_l(\frac{\omega\hat{z}}{2c})J_k(\frac{\omega\hat{z}}{2c}). \qquad (33)$$

This time only modes represented by $\alpha_l^1$ and $\alpha_{-l}^{-1}$ contribute significantly. Keeping these two modes, we get the same secular equation for eigenvalues as Eq.(21) except $\Delta$ is redefined as

$$\Delta = \nu_\beta + l\nu_s - \frac{m}{2} \qquad (34)$$

and the stopband width is

$$\delta\nu_l = \frac{Ne^2C}{8\pi^3\nu_\beta E}|\int_{-\infty}^{\infty} d\omega Z_1^\perp(\omega)J_l^2(\frac{\omega\hat{z}}{2c})|. \qquad (35)$$

For the broad band impedance

$$Z_1^\perp(\omega) = \frac{2cR_0}{b^2\omega_0}|\frac{\omega_0}{\omega}|^{3/2}[sgn(\omega)+i], \qquad (36)$$

where $b$ is beam pipe radius, the stopband width is

$$\delta\nu_l = \frac{\sqrt{2}}{\pi^2\Gamma^2(\frac{1}{4})}\frac{\Gamma(l-\frac{1}{4})}{\Gamma(l+\frac{5}{4})}\frac{\Upsilon}{\nu_\beta}, \qquad (37)$$

with

$$\Upsilon = \frac{eIR_0}{E}\frac{\hat{z}^{1/2}C^{3/2}}{b^2}. \qquad (38)$$

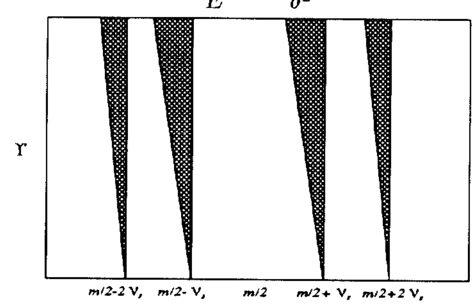


FIG. 2. The transverse stopbands. The shaded regions are unstable. Resonance structures are displayed by this diagram.

## IV. REFERENCES


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