# Longitudinal Head-Tail Instability in a Non-Harmonic Potential Well

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*Abstract*

A perturbation technique is developed that can be applied to study the collective instability problem when the unperturbed system is not described by a simple harmonic oscillator. The longitudinal head-tail instability effect is well studied as applications of this technique. Applications of the longitudinal head-tail instability effects to the CERN SPS and the SSC are included.

## I. INTRODUCTION

The ideal motion of a single particle in a circular accelerator is that of a simple harmonic oscillator. The conventional theory of collective instabilities is developed by imposing the perturbation of collective wake forces on the simple harmonic system.

However, when the new collective longitudinal instability was observed in the CERN SPS[1], the analyses suggested the "longitudinal chromaticity" playing a role. Drawing analogy to the transverse case where the betatron chromaticity causes the head-tail instability, this new instability was named longitudinal head-tail (LHT) instability. The theoretical existence of the LHT instability was pointed out by Hereward[2]; it results from the non-simple-harmonic nature of the system when the longitudinal chromaticity effect is considered. To study the LHT instability, the conventional theory does not suffice because it treats only the simple-harmonic case.

In this paper we develop a new formalism that extends the conventional approach to the non-simple-harmonic Hamiltonian system. The LHT instability is studied as an application to demonstrate the technique. By using the water-bag particle distribution model, it is possible to solve the problem exactly and obtain the growth rates for the various collective modes (the dipole, quadrupole, sextupole modes, etc). Although not discussed below, the potential-well distortion, as well as its effects on collective instabilities, can also be studied with this technique.

## II. MECHANISM OF THE LONGITUDINAL HEAD-TAIL INSTABILITY

The LHT instability, like its well-known transverse counterpart, the transverse head-tail instability, is a single bunch effect. The mechanisms of these instabilities are quite similar. They are caused by a dependence of the accumulated betatron or synchrotron phase on the longitudinal position of the particle, coupled with a perturbation due to the collective wake forces.

However, the situation is more subtle in the longitudinal case since longitudinal position $z$ and $\delta$ are the dynamic variables which describe the particle motion. The analysis of this problem is therefore more involved.

Consider a circular accelerator whose slippage factor $\eta$ contains a higher order term in $\delta$, i.e., $\eta = \eta_0(1 + \frac{3}{2}\epsilon\delta)$, where $\eta_0$ is the leading contribution of the momentum slippage factor, and $\epsilon$ is a parameter that specifies the strength of the higher order contribution. The unperturbed equations of motion of a single particle are given by

$$\frac{dz}{ds} = -\eta_0\delta(1 + \frac{3}{2}\epsilon\delta), \ \frac{d\delta}{ds} = \frac{\omega_s^2}{\eta_0 c^2}z; \qquad (1)$$

where $s$ is the longitudinal coordinate along the accelerator circumference, and $\omega_s$ is the unperturbed synchrotron oscillation frequency for small amplitudes.

The coefficient $\epsilon$ describes the deviation from the simple harmonicity of the system. We consider small $\epsilon$ so that $|\epsilon\delta| \ll 1$. The motion of single particle in the $z$-$\delta$ space follows a constant Hamiltonian contour. One such contour is shown in Fig. 1.

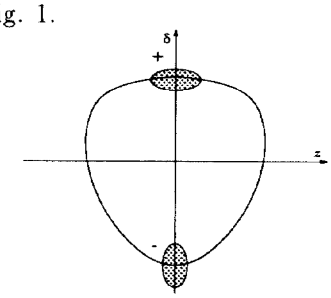


FIG. 1. The phase space trajectory due to the non-simple-harmonic Hamiltonian. The case shown is for $\epsilon > 0$.

The main effect of a non-vanishing $\epsilon$ is that it has introduced an asymmetry between the upper and the lower halves of the phase plane. As the beam bunch executes its dipole oscillation in this deformed phase space, the shape of the phase space area occupied by the bunch varies, although its area is conserved. The bunch shape at two instances (marked by + and −) are shown as shaded areas in Fig.1. In particular, the bunch lengths $\hat{z}_+$ and $\hat{z}_-$ at the two instances are related by the Liouville theorem

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according to

$$\frac{\hat{z}_-}{\hat{z}_+} = \frac{|\frac{dz}{ds}_-|}{|\frac{dz}{ds}_+|} = \frac{-\delta_-(1+\frac{3}{2}\epsilon\delta_-)}{\delta_+(1+\frac{3}{2}\epsilon\delta_+)} \approx \frac{1-\epsilon\delta_0}{1+\epsilon\delta_0} \approx \frac{1+\epsilon\delta_-}{1+\epsilon\delta_+}, \tag{2}$$

where $\delta_0 = \sqrt{2H_0}/|\eta_0|$, and $\delta_\pm \approx \pm\delta_0 - \frac{1}{2}\epsilon\delta_0^2$ are the values of $\delta$ at the + and − locations. We conclude from Eq.(2) that, to first order in $|\epsilon\delta_0|$, the bunch length is modulated according to

$$\hat{z} \propto 1 + \epsilon\delta \tag{3}$$

as the bunch executes the dipole oscillation in the phase space. Next we introduce the effect of the collective wake fields. Since the energy loss of the beam bunch depends on the bunch length, the bunch energy loss is also modulated by the same factor of Eq.(3). Adding the energy loss term to Eq.(1),

$$\frac{d\delta}{ds} \approx \frac{\omega_s^2}{\eta_0 c^2}z + \epsilon\frac{\hat{z}}{NEC}\frac{d\Delta\varepsilon}{d\hat{z}}\delta, \tag{4}$$

where $N$ is number of particles per bunch, $E$ is the particle energy, $C$ is machine circumference, and $\Delta\varepsilon$ is the bunch energy loss per turn(in our convention, $\Delta\varepsilon < 0$). we have kept only its leading contribution to first order in $\delta$.

The new equations of motion represent a system with growth (or damping if negative) rate:

$$\tau^{-1} = \epsilon\frac{c\hat{z}}{2NEC}\frac{d\Delta\varepsilon}{d\hat{z}}. \tag{5}$$

The Eq.(5) was first obtained in Ref.[1].

## III. PERTURBATION APPROACH

From the previous section we knew that to study the LHT instability, we will have to consider an unperturbed system which is described by a non-simple-harmonic Hamiltonian. For such a system, the conventional method of using polar coordinates(the action-angel variables) no longer applies. The technique we develop in this paper is to introduce a new pair of dynamical variables: the Hamiltonian $H$ itself and another variable $Q$ which assumes the role of the time variable. The advantage of using the new variables is we only need to deal with one complicated variable $Q$. This point will become clear in the later derivation. Having introduced the new dynamical variables, the procedure we use to solve the Vlasov equation then follows basically the conventional treatment.

We start with a general situation when the accelerator is described by a Hamiltonian $H(q,p;s)$. The beam distribution is given by

$$\psi(q,p;s) = \psi_0(H_0) + \psi_1(q,p)e^{-i\Omega s/c}. \tag{6}$$

The unperturbed system, $\psi_0$ must be a function of unperturbed $H_0 = \frac{1}{2}p^2[1+f(p)] + \frac{\omega_s^2}{2c^2}q^2$ alone.

$$H(q,p;s) = \frac{1}{2}p^2[1+f(p)] + \frac{\omega_s^2}{2c^2}q^2 - \frac{\eta_0 e}{EC}\int_{-\infty}^{q} V_1(q')dq' e^{-i\Omega s/c}, \tag{7}$$

where the function $f(p)$ represents a small deviation of the system from simple harmonicity, particularly for the system described by Eq.(1) for which $f(p) = -\frac{\epsilon}{\eta_0}p$. and $V_1$ is the retarding wake voltage per turn induced by $\psi_1$ and is related to the longitudinal impedance $Z_0^{\|}(\omega)$ according to

$$V_1 = \frac{e}{2\pi}\int_{-\infty}^{\infty} d\omega Z_0^{\|}(\omega)e^{i\omega q/c}\int_{-\infty}^{\infty} dq' e^{-i\omega q'/c}\int_{-\infty}^{\infty} dp\,\psi_1(q',p), \tag{8}$$

Here the potential-well distortion effect, which is not of interest in the present study, has been ignored. In writing down Eq.(8), we have also ignored multi-turn wake effects.

We now introduce new canonical variables $(Q, H_0)$, where

$$Q = -\int_0^p \frac{\partial q(H_0,p')}{\partial H_0}dp'. \tag{9}$$

The advantage of using $H_0$ as dynamical variable lies in the fact that $\psi_0$ depends on $H_0$ only.

Notice the period of the motion of a particle is $\Phi = \oint \frac{\partial q(H_0,p')}{\partial H_0}dp'$. This period depends on the value of $H_0$ for the particle under consideration.

Considering $\psi_1$ is a small quantity, we use the new variables to obtain the linearized Vlasov equation by keeping the first order terms in $\psi_1$,

$$-i\frac{\Omega}{c}\psi_1 + \frac{\partial\psi_1}{\partial Q} + \frac{\eta_0 e}{EC}V_1(q)\frac{\partial\psi_0}{\partial H_0}\frac{\partial H_0}{\partial p} = 0. \tag{10}$$

To solve the Vlasov equation, we first Fourier expand $\psi_1$ as

$$\psi_1 = \sum_{l=-\infty}^{\infty} R_l(H_0)e^{i2\pi lQ/\Phi(H_0)}, \tag{11}$$

where the $l = 0$ term in the summation is to be excluded because it violates the total charge conservation for a given $H_0$. The Fourier expansion is possible because the motion is periodic in $Q$ with period $\Phi$.

For the $l$-th mode (for example, $l = 1$ corresponds to the dipole mode),

$$\begin{aligned}&\left[\Omega - \frac{2\pi lc}{\Phi(H_0)}\right]R_l(H_0) + i\frac{\eta_0 e^2 c}{2\pi EC\Phi(H_0)}\int_{-\infty}^{\infty} d\omega Z_0^{\|}(\omega)\\ &\times\int_0^{\Phi(H_0)} dQ\frac{\partial\psi_0}{\partial H_0}\frac{\partial H_0}{\partial p}\exp\left[i\omega\frac{q(Q,H_0)}{c}\right]\\ &\times\int_{-\infty}^{\infty} dH_0'\int_0^{\Phi(H_0')} dQ'\exp\left[-i\omega\frac{q(Q',H_0')}{c}\right]\\ &\times\sum_{l'=-\infty}^{\infty} R_{l'}(H_0')\exp\left[i2\pi l'\frac{Q'}{\Phi(H_0')} - i2\pi l\frac{Q}{\Phi(H_0)}\right] = 0.\end{aligned} \tag{12}$$

For a general equilibrium distribution $\psi_0$ (Gaussian, for example), the analysis to solve Eq.(12) is involved. Pursuing along this line would yield the radial modes of the collective oscillation. For one simple beam distribution, the water-bag model, however, the radial modes degenerate and the equation can be solved analytically. In the following, we will assume that the unperturbed beam has a water-bag distribution

$$\psi_0(H_0) = \frac{N}{\int_0^{\hat{H}} \Phi(H_0) dH_0} \Theta(\hat{H} - H_0), \quad (13)$$

where $\Theta$ is the step function. $\hat{H} = \omega_s^2 \hat{z}^2 / 8c^2$.

Since any perturbation of a water-bag distribution has to occur around the edge of the bag, we have

$$R_l(H_0) \propto \delta(H_0 - \hat{H}). \quad (14)$$

After adopting the water-bag model, Eq.(12) can be simplified. Also, the coupling among the different modes with $l' \neq l$ are neglected. The validity of this approximation assume the mode frequency shifts are small compared with $2\pi c/\Phi \approx \omega_s$.

We further define an angular variable $\theta$ according to

$$q = \frac{c}{\omega_s}\sqrt{2H_0}\cos\theta, \; p\sqrt{1 - \frac{\epsilon}{\eta_0}p} = \sqrt{2H_0}\sin\theta. \quad (15)$$

In terms of the new variable $\theta$, Eq.(12) can be written as

$$\Omega^{(l)} - \frac{l\omega_s}{< G >} - i\frac{\eta_0 N e^2 c^2}{2\pi^3 E C \omega_s \hat{z} < G >} \int_{-\infty}^{\infty} d\omega Z_0^{\|}(\omega)$$

$$\times \int_0^{2\pi} d\theta \sin\theta \exp\left[ i\frac{\omega\hat{z}}{2c}\cos\theta + il\frac{\int_0^{\theta} G(\theta'')d\theta''}{< G >} \right]$$

$$\times \int_0^{2\pi} d\theta' G(\theta') \exp\left[ -i\frac{\omega\hat{z}}{2c}\cos\theta' - il\frac{\int_0^{\theta'} G(\theta'')d\theta''}{< G >} \right] = 0. \quad (16)$$

where

$$< G > = \frac{1}{2\pi}\int_0^{2\pi} G(\theta)d\theta, \; G(p) = \frac{\sqrt{1 - \frac{\epsilon}{\eta_0}p}}{1 - \frac{3}{2}\frac{\epsilon}{\eta_0}p}. \quad (17)$$

If the non-harmonicity is weak, we assume $|\epsilon\sqrt{2\hat{H}}/\eta_0| \ll 1$ and keep the first order terms in $\epsilon$ to obtain the mode frequency

$$\Omega^{(l)} = l\omega_s + i\frac{\eta_0 N e^2 c^2}{2\pi^3 E C \omega_s \hat{z}}(A + \frac{\epsilon}{2\eta_0}\frac{\omega_s \hat{z}}{c}B), \quad (18)$$

where

$$A = \frac{8\pi^2 c}{\hat{z}} l \int_{-\infty}^{\infty} d\omega \frac{Z_0^{\|}(\omega)}{\omega} J_l^2(\frac{\omega\hat{z}}{2c}),$$

$$B = -\frac{32\pi^2 c^2}{\hat{z}^2} l^2 \int_{-\infty}^{\infty} d\omega \frac{Z_0^{\|}(\omega)}{\omega^2} [\frac{\omega\hat{z}}{2c} J_l(\frac{\omega\hat{z}}{2c}) J_{l+1}(\frac{\omega\hat{z}}{2c}) + (1-l) J_l^2(\frac{\omega\hat{z}}{2c})] \quad (19)$$

$A$ is purely imaginary and $B$ is real. If $\epsilon = 0$, only the $A$ coefficient plays a role; the result describes the solution of the conventional longitudinal instability problem. In particular, the fact that $A$ is purely imaginary means the mode frequency $\Omega$ is real, and the beam is always stable. This is a well-known result[5][6] when mode coupling and multi-turn effects are ignored, as is presently assumed. If $\epsilon \neq 0$, the $B$ term also contributes to the mode frequency $\Omega$. This contribution, being imaginary, is the cause of the LHT instability. The instability growth rate is

$$\tau_l^{-1} = \epsilon\frac{4Ne^2c^2}{\pi E C \hat{z}} l^2 \int_{-\infty}^{\infty} dx \frac{Re Z_0^{\|}(\frac{2c}{\hat{z}}x)}{x^2}[x J_l(x) J_{l+1}(x) + (1-l) J_l^2(x)]. \quad (20)$$

For the case $l = 1$, we can recover the result given by the simple physical picture. Eq.(5) applies to dipole mode only, while Eq.(20) is valid for arbitrary $l$

For the CERN SPS collider, we estimate that[7] the growth time $\tau$ is 5.4 s for the diffraction broad band impedance model. The observed growth rate is $5 \sim 6$ s[1]. For SSCL machines, LEB, MEB, HEB and collider, the growth time are found to be $7.0 \times 10^4$ s, $3.4 \times 10^3$ s, 32 s and 1.2 s, respectively.

The LHT instabilities tend to play a more important role in the lower energy accelerators, particularly those operated close to transition. In all cases studied, however, the LHT instability does not constitute a serious limit on beam intensities.

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