Computation of Longitudinal Bunched Beam Instability Thresholds

R. Baartman
TRIUMF, 4004 Wesbrook Mall, Vancouver, B.C. V6T 2A3, Canada
and
M. D'yachkov
Physics Department, University of British Columbia,
6224 Agricultural Road, Vancouver, B.C. V6T 1Z1, Canada

Abstract

An integral equation derived from the linearized Vlasov equation has been used to find the instability thresholds in the case of space-charge impedance alone for various distribution functions. It has been found that the thresholds for the instability which are caused by the coupling between $m = \pm 1$ azimuthal modes may be obtained analytically for many practically used distributions. Moreover, the criterion determining these thresholds appears to be the same as that for thresholds beyond which no stationary distribution can be found.

I. INTRODUCTION

It has been found that in the case of broad-band or space-charge impedance the stationary distribution changes significantly with intensity, and this should be taken into account because the stability thresholds calculated ignoring potential well distortion differ from those obtained in self-consistent calculations [2]. Therefore, the results obtained previously [4] under the assumption of absence of incoherent frequency spread have to be considered critically. However, these results provide us with a clear picture of the physics of the instability and can be used for checking any other new theory.

It has been shown in ref. 4 that the problem of determining the $m = \pm 1$ thresholds (as well as others caused by $\pm m$ coupling) in the absence of synchrotron frequency spread can be formulated as an eigenvalue problem for the Fourier components of the line density. Moreover, analytical expressions for matrix elements for some specific distributions have been found [4].

II. INTEGRAL EQUATION

We normalize the longitudinal coordinate q such that the Hamiltonian of the particle is

$$H(p,q) = \frac{p^2}{2} + V(q), \tag{1}$$

where p is the longitudinal momentum and $V(q)$ is a potential which we assume to be symmetric.

In the case of space-charge impedance, the self-force is proportional to the derivative of the line density and we also can define an intensity parameter I so that

$$F_{SC} = -I\frac{d\lambda}{dq} \text{ and } V = V_0 + I\,\lambda(q), \tag{2}$$

where $\lambda(q)$ is the line density. I may have either sign: for space-charge it is positive below transition and negative above.

The Vlasov equation can be written in terms of p an q, or, more conveniently, in action-angle variables as

$$\frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial\theta}\dot{\theta} + \frac{\partial\psi}{\partial J}\dot{J} = 0, \tag{3}$$

where $\dot{J} = -\partial H/\partial\theta$ and $\dot{\theta} = \partial H/\partial J$.

It can be shown that the stationary distribution ψ_0 is a function of J only. We therefore look for a solution in the form $\psi = \psi_0(J) + \psi_1(J,\theta,t)$, where ψ_1 is a small perturbation. The Hamiltonian of the system can then be written as

$$H(J,\theta,t) = H_0(J) + I\,\lambda_1(q(J,\theta),t), \tag{4}$$

where $\lambda_1(q,t) = \int \psi_1\, dp$. Dropping terms of second order and taking into account that $dH_0/dJ = \omega(J)$, the linearized Vlasov equation becomes

$$\frac{\partial\psi_1}{\partial t} + \omega(J)\frac{\partial\psi_1}{\partial\theta} - I\frac{\partial\lambda_1}{\partial\theta}\frac{d\psi_0}{dJ} = 0. \tag{5}$$

It turns out to be more convenient to take ψ_0 and ω to be a functions of $\epsilon = H(p,q)$ instead of J:

$$\psi_0'(\epsilon) \equiv \frac{d\psi_0}{d\epsilon} = \frac{1}{\omega(J)}\frac{d\psi_0}{dJ}. \tag{6}$$

We look for a solution in the form $\psi_1 = fe^{\nu t}$ (and $\lambda_1 = ge^{\nu t}$) Then with the definition $\Omega(\epsilon) = \nu/\omega(\epsilon)$, we get

$$\Omega f + \frac{\partial f}{\partial\theta} - I\psi_0'(\epsilon)\frac{\partial g}{\partial\theta} = 0. \tag{7}$$

The periodic solution $f(\epsilon,\theta) = f(\epsilon,\theta+2\pi)$ is

$$f(\epsilon,\theta) = I\frac{\psi_0'(\epsilon)}{e^{2\pi\Omega}-1}\int_\theta^{\theta+2\pi} e^{\Omega(\theta-\theta')}\frac{\partial g}{\partial\theta'}d\theta'. \tag{8}$$

Note that although this result is formally equivalent to that given by Krinsky and Wang [3, eqn. 3.18], it differs in the sense that the present treatment is a perturbation about the stationary case which includes the space-charge impedance: in ref. 3, the stationary induced potential is ignored.

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Integrating eqn. 8 by parts we get

$$f(\epsilon,\theta) = I\psi_0'(\epsilon)g(q) - I\psi_0'(\epsilon)\frac{\Omega}{e^{2\pi\Omega}-1}\int_\theta^{\theta+2\pi} e^{\Omega(\theta-\theta')}g(q')d\theta'. \quad (9)$$

Integrating eqn. 9 over the momentum and taking into account that $g(q) = \int f(\epsilon,\theta)dp$, we have finally

$$g(q)\left[1 - I\Lambda'(V)\right] = I\int_{-\infty}^{\infty} dp\psi_0'(\epsilon)\frac{\Omega}{e^{2\pi\Omega}-1}\int_\theta^{\theta+2\pi} e^{\Omega(\theta-\theta')}g(q')d\theta', \quad (10)$$

where $\Lambda'(V) = d\Lambda/dV$, and

$$\Lambda(V) = \sqrt{2}\int_V^\infty \frac{\psi_0(\epsilon)d\epsilon}{\sqrt{\epsilon - V}} \quad (11)$$

is an auxiliary function for $\psi_0(H)$ which we introduced previously [1] in connection with finding stationary distributions.

Eqn. 11 is non-linear with respect to ν and therefore is not easy to solve in general. In the special case $\nu \to 0$, however, we have the simple result

$$g(q)\left[1 - I\Lambda'(V)\right] = -\frac{I}{\pi}\int_{-\infty}^{\infty}\psi_0'(\epsilon)dp\int_0^\pi g(q')d\theta'. \quad (12)$$

The $\nu \to 0$ limit can be thought of as coupling between $\pm m$ azimuthal modes [4]. Since the dipole mode $m = \pm 1$ is the lowest order antisymmetric eigenmode $g(q) = -g(-q)$, the integral in eqn. 12 vanishes and we find

$$g(q)\left[1 - I\Lambda'(V)\right] = 0. \quad (13)$$

III. MODE-COUPLING THEORY

A different method to determine thresholds of longitudinal bunched beam stability in the presence of space-charge was used in ref. 4. The thresholds corresponding to coupling between $\pm m$ azimuthal modes can be formulated as an eigenvalue problem for the Fourier components of the line density. To compare the two techniques we have chosen the family of distributions defined by

$$\psi_0(\epsilon) = \frac{1}{\sqrt{2\pi}}\left[1 - \alpha(1-\epsilon)\right]e^{-\epsilon} \quad (14)$$

This family is very convenient because it contains both the 'gaussian' ($\alpha = 0$) and 'hollow-gaussian' ($\alpha = 1$) cases, and because the matrix elements of the eigenvalue problem can be found analytically. The eigenvalue problem can be written in the form

$$\mu\, g_k = \sum_{l=-\infty}^{\infty} H_{kl}\, g_l, \quad (15)$$

where $g(q) = \sum g_k e^{ikq}$ and the matrix elements are

$$H_{kl} = -\frac{1-\alpha}{\sqrt{2\pi}}\left[1 - \bar{I}_0(kl)\right] - \frac{\alpha}{\sqrt{2\pi}}\left\{-\frac{k^2+l^2}{2}\left[1 - \bar{I}_0(kl)\right] + kl\left[1 - \bar{I}_1(kl)\right]\right\}. \quad (16)$$

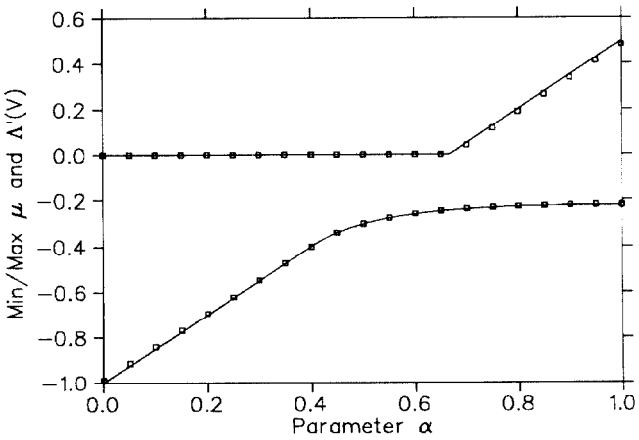

Figure 1: Extreme eigenvalues of eqn. 15 (plotted as squares) and extreme values of $\Lambda'(V)$ (solid lines).

Here $\bar{I}_m(z) = e^{-z}I_m(z)$ is the exponentially scaled modified Bessel function. The lowest thresholds can be found from the condition $\hat{\mu}\, I_{\rm th} = 1$, where $\hat{\mu}$ are the extreme eigenvalues (of either sign).

In comparison, we have from eqn. 13 that $1 - \widehat{\Lambda'(V)}\, I_{\rm th} = 0$. With ψ_0 given by eqn. 14, the expression for $\Lambda(V)$ is

$$\Lambda(V) = \left(1 - \frac{\alpha}{2} + \alpha V\right)e^{-V}. \quad (17)$$

Simple analysis allows us to obtain the following expressions for extreme values of $\Lambda'(V)$ for different parameters α.

$$\Lambda'(V)_{\rm min} = \begin{cases} -1 + \frac{3}{2}\alpha & \text{if } 0 < \alpha < \frac{2}{5} \\ -\alpha\exp\left(\frac{1}{\alpha} - \frac{5}{2}\right) & \text{if } \frac{2}{5} < \alpha < 1 \end{cases}$$

$$\Lambda'(V)_{\rm max} = \begin{cases} 0 & \text{if } 0 < \alpha < \frac{2}{3} \\ -1 + \frac{3}{2}\alpha & \text{if } \frac{2}{3} < \alpha < 1 \end{cases} \quad (18)$$

Therefore, if one plots $\hat{\mu}$ and $\widehat{\Lambda'(V)}$ vs. α on the same graph, the two curves should be the same. In order to solve numerically eqn. 15, the matrix was truncated at 40×40. Minimum and maximum eigenvalues obtained in this case are the plotted points in Fig. 1. The solid lines are $\widehat{\Lambda'(V)}$ (18). We can see that the results are in good agreement.

IV. SELF-CONSISTENT CASE

The results discussed in the previous section have been obtained assuming no incoherent synchrotron frequency spread (i.e. $V \propto q^2$). To satisfy this condition in the self-consistent case, the initial potential well should be

$$V_0 = V + I\left[\Lambda(0) - \Lambda(V)\right], \quad (19)$$

where $V = q^2/2$, and $\Lambda(V)$ is given by eqn. 11. The necessary $V_0(q)$ to get a self-consistent stationary phase space distribution with $V(q) = q^2/2$ for the 'hollow-gaussian'

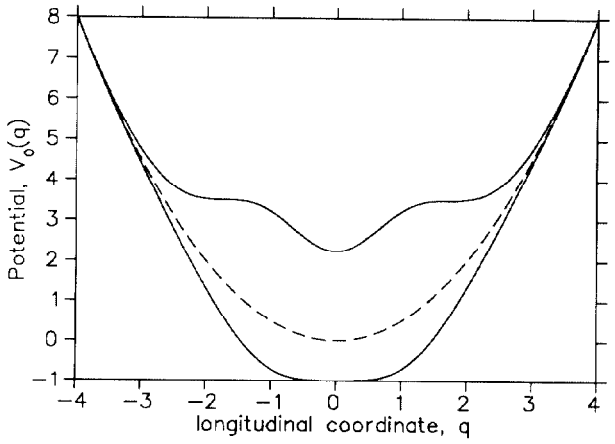

Figure 2: V_0 vs. q at threshold intensities for the 'hollow-gaussian' distribution. $V = q^2/2$ is shown by the dashed curve.

distribution $\psi_0 = \epsilon e^{-\epsilon}$ at threshold intensities are shown in Fig. 2. The upper curve shows $V_0(q)$ at the *positive mass* threshold and the bottom one at the *negative mass* threshold. As one can see these shapes are far from 'sinusoidal' or 'harmonic'.

Realistically, of course, V_0 is harmonic and $V(q)$ is distorted by space-charge. In this case we can find the thresholds beyond which no stationary distribution exists [1]. Surprisingly, the stationarity criterion found in ref. 1 is the same as we obtained for thresholds corresponding to $m = \pm 1$ mode-coupling! This means that in the case when V_0 is a harmonic potential, $m = \pm 1$ modes do not couple.

It is necessary to mention that this criterion is valid for any $V_0(q)$ for which $dV_0/d(q^2) \neq 0$ (i.e. no local minima). This analysis can be extended to the case when $V_0(q)$ is not symmetric. Unfortunately, in this case we can't use the symmetry of the eigenfunctions to determine the thresholds as has been done earlier in this paper. Numerical solution of the integral equation is required. This has been done for several cases of ψ_0 and the results are consistent with the same threshold, namely $I\Lambda'(V) = 1$. However, no formal proof of the universality of this criterion has been found.

V. CONCLUSION

A simple criterion for the thresholds given by vanishing of the real part of the eigenfrequency (coupling between $m = \pm 1$ azimuthal modes) for the bunched beam in the case of space-charge impedance has been derived from the linearized Vlasov equation. The thresholds obtained from this criterion have been found to be in good agreement with the thresholds obtained by the mode-coupling method for the family of distributions which includes 'gaussian' and 'hollow-gaussian' cases. The method described in ref. 4 neglects the potential well distortion and the incoherent synchrotron frequency spread caused by nonlinear space-charge forces. It has been found that when the potential can be approximated by a parabolic one $V_0 = q^2/2$ (or, in general, when $dV_0(q^2)/d(q^2) > 0$), coupling between $m = \pm 1$ modes cannot occur for any stationary distribution.

We also can make a conclusion about the shape of the eigenfunction $g(q)$ at threshold: since the threshold condition (13) is satisfied in general at only one specific point $q_{\rm th}$, $g(q)$ can be non-zero only at $q_{\rm th}$. Indeed, recovering g from the eigenfunction $\{g_k\}$ corresponding to $\mu_n = \widehat{\mu}$ [4], we find a very sharp peak at the point where $I_{\rm th}\,\widehat{\Lambda'(V)} = 1$ and almost zero elsewhere, and the peak becomes sharper with increasing order of the matrix used in eqn. 15.

VI. REFERENCES

[1] M. D'yachkov, R. Baartman *Methods for Finding Stationary Longitudinal Distributions*, Proc. HEACC'92 (Hamburg), p. 1064.

[2] K. Oide and K. Yokoya *Longitudinal Single-Bunch Instability in Electron Storage Rings*, KEK Preprint 90-10 (1990).

[3] S. Krinsky, J.M. Wang *Longitudinal Instability of Bunched Beams Subject to a Non-Harmonic RF Potential*, Particle Accelerators **17**, p. 109 (1985).

[4] R. Baartman, B. Zotter *Longitudinal Stability of Hollow Beams II: Mode-Coupling Theory*, TRIUMF Note TRI-DN-91-K177 (1991).