Calculation of the Bunch Lengthening Threshold

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Abstract

A new analysis of the bunch lengthening instability, based on a single synchrotron mode in a distorted potential well, is presented. The nonlinearity of the wakefield plays a critical role: It distorts the equilibrium density from its Gaussian shape, which results in asymmetric corrections to the Sacherer equation. This modified equation will have unstable eigenmodes when the beam current reaches a threshold value. The calculated threshold agrees very well with our multiparticle simulation for SPEAR parameters [1].

I. INTRODUCTION

The performance of modern synchrotron light sources and circular colliders relies on sustaining very short bunches of high peak currents. The bunch lengthening instability, i.e., the sudden increase of both the bunch length and the energy spread at some threshold current, is a serious concern. Much theoretical work has been aimed at explaining this phenomenon [2]. Most notably, mode coupling theory assumes that low order synchrotron modes couple together at the bunch lengthening threshold. However, there is no cogent evidence from experiments that this has actually occurred. In this paper, we investigate the instability mechanism within a particular synchrotron mode in a distorted potential well.

II. LONGITUDINAL DYNAMICS AND EQUILIBRIUM

The mapping for the longitudinal motion in a ring is [3]:

$$\tau_{n+1} = \tau_n - \alpha T_0 \delta_{n+1}, \tag{1}$$

$$\delta_{n+1} = \delta_n + \frac{\omega_{s0}^2 T_0}{\alpha}\tau_n - \frac{\text{wakefield loss}}{E_0}, \tag{2}$$

where τ_n is the arrival time relative to the synchronous particle in the nth turn, δ_n is its relative energy error, α the momentum compaction factor of the ring, T_0 the revolution time, ω_{s0} the synchrotron oscillation frequency and E_0 the beam energy.

Since the synchrotron tune is small for most rings and the wakefield loss is distributed throughout the ring, we can then approximate the mapping by the differential equations:

$$\frac{d\tau}{dt} = -\alpha\delta, \tag{3}$$

$$\frac{d\delta}{dt} = \frac{\omega_{s0}^2}{\alpha}\tau - \frac{1}{E_0 T_0}F(\tau). \tag{4}$$

Here $F(\tau)$ is the wakefield loss of the particle:

$$F(\tau) = Ne^2 L_0 \int_\tau^{+\infty} d\tau' \rho(\tau') W(\tau' - \tau). \tag{5}$$

Eq. 5 involves integrating the wake left by all other charges in front of the particle under consideration. N is the number of particles in the bunch, L_0 the ring circumference, ρ the particle density of the bunch and W the longitudinal wake function [4].

If we identify τ as the coordinate and δ the momentum, then Eqs. 3 and 4 are the dynamic equations of a harmonic oscillator under the influence of the additional force $F(\tau)$. The Hamiltonian of this system is:

$$H = \frac{\delta^2}{2} + \frac{\omega_{s0}^2\tau^2}{2\alpha^2} - \frac{Ne}{E_0 T_0}\int_0^\tau d\tau' F(\tau'). \tag{6}$$

For electron machines, synchrotron radiation of electrons and its quantum nature provide damping and diffusion in phase space. The resulting equilibrium state will be a Boltzmann distribution:

$$\psi_0(\tau,\delta) \propto \exp(-\frac{H}{\sigma_{\delta 0}^2}), \tag{7}$$

where $\sigma_{\delta 0}$, the equilibrium energy spread, is determined by balancing the radiation damping and the diffusion caused by quantum excitation.

It is very useful to observe that, while the wakefield can change the particle distribution in τ, it cannot change the distribution in δ. It will always be a Gaussian distribution with constant energy spread. Experiments show that

*This work was supported by the U.S. Department of Energy, Division of Nuclear and High Energy Physics.

0-7803-1203-1/93$03.00 © 1993 IEEE

the energy spread increases after the current reaches some threshold value. In our simulations, we also observe that the distribution in δ starts to develop a non-Gaussian deformation at the threshold. All these suggest that the equilibrium is unstable as the current reaches the threshold.

III. Sacherer Equation

We follow the Vlasov-Sacherer [5] approach to investigate the stability of the equilibrium. The phase space distribution function, $\psi(\tau,\delta,t)$, satisfies:

$$\frac{\partial\psi}{\partial t}-\alpha\delta\frac{\partial\psi}{\partial\tau}+\frac{\omega_{s0}^2}{\alpha}\tau\frac{\partial\psi}{\partial\delta}-\frac{eF(\tau)}{E_0T_0}\frac{\partial\psi}{\partial\delta}=0. \tag{8}$$

To introduce the concept of a synchrotron mode, we first neglect the wakefield, i.e., we consider the zero current limit. In polar coordinates, $\tau=r\cos\phi$, $\alpha\delta/\omega_{s0}=r\sin\phi$ and the Vlasov equation reduces to

$$\frac{\partial\psi}{\partial t}+\omega_{s0}\frac{\partial\psi}{\partial\phi}=0. \tag{9}$$

Thus, the equilibrium distribution, ψ_0, depends only on r. The eigenmodes of the time dependent solutions are

$$\psi=R_l(r)\exp(il\phi-i\Omega^{(l)}t), \tag{10}$$

with eigenvalues

$$\Omega^{(l)}=l\omega_{s0}, \qquad l=0,\pm1,\pm2,\cdots. \tag{11}$$

These are called synchrotron modes.

When the current is not zero, different l modes are coupled together by the wakefield. We know from most experiments and simulations that low synchrotron modes are still well separated even when the current reaches the threshold. So we assume the coupling between different synchrotron modes is not critical for the bunch lengthening instability and we will neglect mode coupling henceforth.

Next we keep only the linear term in $F_0(\tau)$:

$$F_0(\tau)=F_0(0)+\frac{dF_0}{d\tau}\tau+O(\tau^2). \tag{12}$$

Here the subscript denotes the wakefield loss from the equilibrium particle density distribution. It is easy to see that all the wakefield effects on the zeroth order Vlasov equation can be incorporated into an incoherent synchrotron frequency shift

$$\omega_{s0}^2\to\omega_s^2[1-\frac{\alpha e}{E_0T_0\omega_{s0}^2}\frac{dF_0}{d\tau}], \tag{13}$$

and a shift of the center of the bunch:

$$\tau\to\tau'=\tau-\tau_0, \tag{14}$$

$$\tau_0=\frac{\alpha e}{E_0T_0\omega_s^2}F_0(0). \tag{15}$$

Thus, ψ_0 is still a function of r:

$$\psi_0(r)\propto\exp(-\frac{r^2}{2\sigma_\tau^2}). \tag{16}$$

If we project the first order Vlasov equation

$$\frac{\partial\psi_1}{\partial t}+\omega_s\frac{\partial\psi_1}{\partial\phi}-\frac{e}{E_0T_0}F_1\frac{\partial\psi_0}{\partial\delta}=0, \tag{17}$$

into a particular synchrotron mode, we obtain the Sacherer equation:

$$(\Omega-l\omega_s)R_l(r)=\frac{Ne^2}{E_0T_0}l\frac{\psi_0'(r)}{r}\int r'dr'G_l(r,r')R_l(r'). \tag{18}$$

Here, the kernel

$$G_l(r,r')=\int d\omega\frac{ImZ(\omega)}{\omega}J_l(\omega r)J_l(\omega r'), \tag{19}$$

is determined by the imaginary part of the impedance $Z(\omega)$. Since G_l is real and symmetric, there are only real eigenvalues. The single mode Sacherer equation does not provide the instability mechanism we are looking for.

IV. Improved Sacherer Equation

The discussion above suggests that it is crucial to keep the nonlinear terms in the wakefield loss $F(\tau)$. The most important consequence is that the wakefield effects cannot be completely incorporated into the incoherent frequency shift and the shift of the center of the bunch. Although ψ_0 is still Gaussian in δ, it is no longer Gaussian in τ. Because of this non-Gaussian deformation, ψ_0 is a function of both r and ϕ: $\psi_0(r,\phi)$.

From Eq. 17, the ϕ dependence of ψ_0 will generate correction terms to the right hand side of the Sacherer equation. Generally, they are not symmetric operators. Since the ϕ dependence in the equilibrium is at least linear in the current, the correction terms to the Sacherer equation are proportional to the square of the current. The improved Sacherer equation will look like:

$$\begin{aligned}(\Omega-l\omega_s)R_l(r) &= I\times\ \text{symmetric operator on } R_l\\ &+\ I^2\times\ \text{asymmetric operator on } R_l.\end{aligned} \tag{20}$$

When the current I is small, we don't expect the asymmetric perturbation to be big enough to push the eigenvalues of the symmetric Sacherer operator into the complex plane. When I reaches some critical value, we expect that this will happen and then the system goes unstable.

V. Parameterization of the Equilibrium

In order to find the corrections to the Sacherer equation, we need to know the functional form of $\psi_0(r,\phi)$. To accomplish this, Fourier expand ψ_0:

$$\psi_0(r,\phi)=f_0(r)+f_1(r)\cos\phi+f_2(r)\cos2\phi+\cdots. \tag{21}$$

We will determine $f_0(r)$, $f_1(r)$, ... by the following scheme:

The equilibrium density distribution is determined by Haissinski [6] equation:

$$\rho(\tau) = A_0 \exp\left(-\frac{\tau^2}{2\sigma_{\tau 0}^2} - I\frac{\alpha e L_0}{\omega_{s0}^2 E_0 \sigma_{\tau 0}^2}\int_0^{+\infty} dt \rho(t+\tau) g(t)\right), \tag{22}$$

where $g(t)$ is given by:

$$g(t) = \int_0^t dt' W(t'). \tag{23}$$

We can solve this equation numerically and compute the first four cummulants of the particle density distribution $\rho(\tau)$: τ_0, average τ; σ_τ, standard deviation; γ_1, normalized skew moment; γ_2, normalized excess moment. Then we approximate the equilibrium by a cummulant expansion. Denote $x = (\tau - \tau_0)/\sigma_\tau$,

$$\rho(\tau) = [1 + \frac{\gamma_1}{6}(x^3 - 3x) + \frac{\gamma_2}{24}(x^4 - 6x^2 + 3)]n(x). \tag{24}$$

Here $n(x)$ is the standardized Gaussian distribution:

$$n(x) = \frac{1}{\sqrt{2\pi}\sigma_\tau}\exp(-\frac{x^2}{2}). \tag{25}$$

Based on this approximated distribution, we have only the following five non-zero terms: f_0, f_1, f_2, f_3, f_4:

$$f_0(r) = [1 + \frac{\gamma_2}{8}(\frac{r^4}{8\sigma_\tau^4} - \frac{r^2}{\sigma_\tau^2} + 1)]f_{00}(r), \tag{26}$$

$$f_1(r) = \frac{\gamma_1}{8}f_{00}(r)\left(\frac{r^3}{\sigma_\tau^3} - 4\frac{r}{\sigma_\tau}\right), \tag{27}$$

$$f_2(r) = \frac{\gamma_2}{48}f_{00}(r)\left(\frac{r^4}{\sigma_\tau^4} - 6\frac{r^2}{\sigma_\tau^2}\right), \tag{28}$$

$$f_3(r) = \frac{\gamma_1}{24}f_{00}(r)\frac{r^3}{\sigma_\tau^3}, \tag{29}$$

$$f_4(r) = \frac{\gamma_2}{192}f_{00}(r)\frac{r^4}{\sigma_\tau^4}. \tag{30}$$

Here

$$f_{00}(r) = \frac{\alpha}{2\pi\omega_s\sigma_\tau^2}\exp(-\frac{r^2}{2\sigma_\tau^2}). \tag{31}$$

We are now in a position to write down the explicit form of the improved Sacherer equation:

$$(\Omega - l\omega_s)R_l(r) = \frac{Ne^2}{E_0T_0}l\frac{g_1(r)}{r}\int r'dr'G_l(r,r')R_l(r') + \frac{Ne^2}{2E_0T_0}\sum_{n=2}^{5} g_n(r)\int r'dr'G_l^{(n)}(r,r')R_l(r'). \tag{32}$$

Here, the asymmetric kernels are given by

$$G_l^{(n)} = \frac{1}{i^n}\int d\omega Z(\omega)\left(J_{l-n}(\omega r) - (-1)^n J_{l+n}(\omega r)\right)J_l(\omega r'), \tag{33}$$

and the functions g are related to functions f by

$$g_n(r) = \frac{r}{2\sigma_\tau^2}\left(f_{n+1}(r) - f_{n-1}(r)\right). \tag{34}$$

The difference between the approximated distribution and the numerically obtained distribution is very small for the SPEAR parameters. For other impedances and equilibrium bunch lengths, the model expansion Eq. 24 may not be valid. The fundamental point is unchanged: nonlinearity of the wakefield loss leads to instability.

VI. Results and Conclusion

We have calculated the threshold current for SPEAR parameters. Since we are most interested in the instability mechanism behind the bunch lengthening, we linearized the RF bucket and neglected the multi-turn wakefield. Rather than comparing the calculated threshold current with experimental results, we compared it with our multiparticle simulation. Approximating the SPEAR impedance by a $Q = 1$ resonator, the simulation gives a threshold current of around 45 mA. Our calculation gives 50 mA. The first unstable modes in our calculation are the dipole mode and quadrupole mode. In the experiment, the quadrupole mode is observed to be the first unstable synchrotron mode.

In summary, we investigated the bunch lengthening instability of an uncoupled synchrotron mode in a distorted potential well. Without the nonlinearity of the wakefield, the Sacherer equation does not have an unstable eigenmode. The nonlinearity of the wakefield gives asymmetric correction terms to the Sacherer equation. The improved Sacherer equation is unstable when the beam current reaches a threshold value. Linear theory gives a threshold current very close to the simulation result and identifies the same unstable mode as seen in the experiment. Future work includes extending the comparison to other rings and impedances and taking into account the nonlinearity of the RF bucket.

Acknowledgments

We would like to thank Dr. Alex Chao, Dr. Senyu Chen and Mr. Bo Chen for stimulating discussions.

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