RF noise revisited: the effect of coherence.

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Abstract

The theory of RF noise-induced diffusion is reconsidered to account for the coherent nature of the noise, when all particle are affected by the same random force. The resulting small-scale spatial fluctuations ("microstructure") of the beam density are analyzed. The power spectrum of the fluctuations is calculated.

I. INTRODUCTION

The theory of the longitudinal density diffusion due to RF noise was studied by several authors /2-5/. That analysis however ignored the fact that all the particles in the bunch(es) are affected by the same realization of the random force (what can be termed a coherent random driving). Instead, a conventional Brownian motion problem, with statistically independent forces for different particles, was considered. In the present paper, we derive the condition when this substitution is justified in the zero-order approximation. We investigate as well the first order effect, which turns out to be the formation of the "microstructure", i.e. the short-wavelength spatial fluctuations, of the beam density. More details of the calculations can be found in /6/.

II. MODEL.

We consider the general form of the Hamiltonian of nonlinear oscillator with a random driving:

$$H = \frac{p^2}{2} + g(q) + h(q)\xi(t) \qquad (1)$$

where $g(q)$ is an arbitrary nonlinear potential and $\xi(t)$ is, for simplicity yet without loss of generality, chosen to be the white noise, i.e.

$$\langle \xi(t)\xi(t')\rangle = \delta(t-t') \qquad (2)$$

The evolution of the density distribution is governed by the Vlasov equation:

$$\frac{\partial f}{\partial t} - \left(\frac{\partial g}{\partial q} + \frac{\partial h}{\partial q}\xi(t)\right)\frac{\partial f}{\partial p} + p\frac{\partial f}{\partial q} = 0 \qquad (3)$$

The statistical properties of the fluctuating quantity f are appropriately defined by the ensemble average of the distribution function:

$$\bar{f}(p,q,t) = \langle f(p,q,t)\rangle_{\{\xi\}} \qquad (4)$$

and the correlation function of the density fluctuations in the adjacent phase space points:

$$K(p,q,\bar{p},\bar{q},t) = \langle (f(p,q,t)-\bar{f}(p,q,t))(f(\bar{p},\bar{q},t)-\bar{f}(\bar{p},\bar{q},t))\rangle_{\{\xi\}} \qquad (5)$$

We limit ourselves with considering only the same-time correlator K and study therefore only the spatial, but not the time, correlation properties of the fluctuations.

The further analysis will be using the action-angle variables J, Ψ of the unperturbed ($h(q)=0$) Hamiltonian (1), which will be assumed to be known. The perturbed Hamiltonian H in these variables has the form:

$$H = H_0(J) + V(J,\Psi)\xi(t) \qquad (6)$$

where $V(J,\Psi) = h(q(J,\Psi))$ and $H_0(J)$ are known functions.

III. EVOLUTION EQUATIONS.

Both the average density $\bar{f}$ and the correlator K are evolving in time. We will derive the evolution equations for both quantities using basically the conventional techniques of the theory of stochastic differential equations /1/. It had been shown previously /2-4/ that the evolution of the average density obeys the Fokker-Planck equation. The evolution of the density fluctuations however has never been studied.

In the action-angle variables, the mean and the correlator are given by: $\bar{f} = \bar{f}(J,\Psi,t)$ and $K = K(J,\Psi,\bar{J},\bar{\Psi},t)$. We will also use the notation $\bar{\Psi} = \Psi + \varphi$ and compressed

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notations for the phase space coordinates $(1) = x_{1i} = (J, \Psi)$ and $(2) = x_{2i} = (\bar{J}, \bar{\Psi})$. Taking the differentially small time increment Δt, one obtains the derivatives of the average density:

$$\frac{\partial \bar{f}(1)}{\partial t} = \lim_{\Delta t \to 0} \frac{\Delta f(1)}{\Delta t} \tag{7}$$

and the correlator:

$$\begin{aligned}\frac{\partial K}{\partial t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} (&\langle \Delta f(1) f(2) \rangle + \langle f(1) \Delta f(2) \rangle \\ &+ \langle \Delta f(1) \Delta f(2) \rangle - \bar{f}(1) \langle \Delta f(2) \rangle - \\ &- \bar{f}(2) \langle \Delta f(1) \rangle - \langle \Delta f(1) \rangle \langle \Delta f(2) \rangle)\end{aligned} \tag{8}$$

where the increment of the density $\Delta f = f(t + \Delta t) - f(t)$ can be expressed, due to the conservation of the phase-space density, as

$$\Delta f = \frac{\partial f}{\partial x_i} \Delta x_i + \frac{1}{2} \frac{\partial^2 f}{\partial x_i \partial x_k} \Delta x_i \Delta x_k \tag{9}$$

The increments of the phase-space variables Δx_i in time Δt can be obtained from the stochastic equations of motion. The second order terms in Δx were kept because of the properties of the white noise, where the average of quadratic terms $\Delta x_i \Delta x_j$ produces a linear in Δt terms.

The evolution equation for $\bar{f}$ after the substitution of the averages in equation (8) that can be calculated by standard techiques from the relation (9) and equations of motion, becomes the conventional Fokker-Plank equation:

$$\begin{aligned}\frac{\partial \bar{f}}{\partial t} &= \left(\frac{\partial^2 V}{\partial J \partial \Psi} \frac{\partial V}{\partial J} - \frac{\partial^2 V}{\partial J^2} \frac{\partial V}{\partial \Psi} - \omega(J) \right) \frac{\partial \bar{f}}{\partial \Psi} + \\ &+ \left(\frac{\partial^2 V}{\partial \Psi^2} \frac{\partial V}{\partial J} - \frac{\partial^2 V}{\partial \Psi \partial J} \frac{\partial V}{\partial \Psi} \right) \frac{\partial \bar{f}}{\partial J} + \frac{1}{2} \left(\frac{\partial V}{\partial J} \right)^2 \frac{\partial^2 \bar{f}}{\partial \Psi^2} + \\ &+ \frac{1}{2} \left(\frac{\partial V}{\partial \Psi} \right)^2 \frac{\partial^2 \bar{f}}{\partial J^2} - \frac{\partial V}{\partial \Psi} \frac{\partial V}{\partial J} \frac{\partial^2 \bar{f}}{\partial \Psi \partial J}\end{aligned} \tag{10}$$

where $V = V(J, \Psi)$. For the correlator K, one obtains an evolution equation that is coupled to the mean $\bar{f}$:

$$\begin{aligned}\frac{\partial K}{\partial t} &= \left(\frac{\partial^2 V}{\partial J \partial \Psi} \frac{\partial V}{\partial J} - \frac{\partial^2 V}{\partial J^2} \frac{\partial V}{\partial \Psi} - \omega(J) \right) \frac{\partial K}{\partial \Psi} \\ &+ \left(\frac{\partial^2 V}{\partial \Psi^2} \frac{\partial V}{\partial J} - \frac{\partial^2 V}{\partial \Psi \partial J} \frac{\partial V}{\partial \Psi} \right) \frac{\partial K}{\partial J} + \frac{1}{2} \left(\frac{\partial V}{\partial J} \right)^2 \frac{\partial^2 K}{\partial \Psi^2} \\ &+ \frac{1}{2} \left(\frac{\partial V}{\partial \Psi} \right)^2 \frac{\partial^2 K}{\partial J^2} - \frac{\partial V}{\partial \Psi} \frac{\partial V}{\partial J} \frac{\partial^2 K}{\partial \Psi \partial J} \\ &+ \left(\frac{\partial^2 \bar{V}}{\partial \bar{J} \partial \bar{\Psi}} \frac{\partial \bar{V}}{\partial \bar{J}} - \frac{\partial^2 \bar{V}}{\partial \bar{J}^2} \frac{\partial \bar{V}}{\partial \bar{\Psi}} - \omega(\bar{J}) \right) \frac{\partial K}{\partial \bar{\Psi}} \\ &+ \left(\frac{\partial^2 \bar{V}}{\partial \bar{\Psi}^2} \frac{\partial \bar{V}}{\partial \bar{J}} - \frac{\partial^2 \bar{V}}{\partial \bar{\Psi} \partial \bar{J}} \frac{\partial \bar{V}}{\partial \bar{\Psi}} \right) \frac{\partial K}{\partial \bar{J}} + \frac{1}{2} \left(\frac{\partial \bar{V}}{\partial \bar{J}} \right)^2 \frac{\partial^2 K}{\partial \bar{\Psi}^2} \\ &+ \frac{1}{2} \left(\frac{\partial \bar{V}}{\partial \bar{\Psi}} \right)^2 \frac{\partial^2 K}{\partial \bar{J}^2} - \frac{\partial \bar{V}}{\partial \bar{\Psi}} \frac{\partial \bar{V}}{\partial \bar{J}} \frac{\partial^2 K}{\partial \bar{\Psi} \partial \bar{J}} \\ &+ \frac{1}{2} \frac{\partial V}{\partial \Psi} \frac{\partial \bar{V}}{\partial \bar{\Psi}} \frac{\partial^2 K}{\partial J \partial \bar{J}} - \frac{1}{2} \frac{\partial V}{\partial \Psi} \frac{\partial \bar{V}}{\partial \bar{J}} \frac{\partial^2 K}{\partial J \partial \bar{\Psi}} \\ &- \frac{1}{2} \frac{\partial V}{\partial J} \frac{\partial \bar{V}}{\partial \bar{\Psi}} \frac{\partial^2 K}{\partial \Psi \partial \bar{J}} + \frac{1}{2} \frac{\partial V}{\partial J} \frac{\partial \bar{V}}{\partial \bar{J}} \frac{\partial^2 K}{\partial \Psi \partial \bar{\Psi}}\end{aligned} \tag{11}$$

where $\bar{V} = V(\bar{J}, \bar{\Psi})$.

On the long time scale, or similarly in the small noise / fast oscillations regime, one can average the dependence of all quantities along the unperturbed trajectories $J = \text{const}$, $\Psi = \omega(J)t$. This approximation is well known under the name of "averaging of fast-oscillating variables" in the theory of Fokker-Plank equations (see, e.g. /1/). It was also implemented in the previous studies of the average density diffusion in the papers /2-4/. The resulting averaged Fokker-Plank equation becomes the well known diffusion equation, /2-4/

$$\frac{\partial \bar{f}}{\partial t} = \frac{\partial}{\partial J} \left(D_J(J) \frac{\partial \bar{f}}{\partial J} \right) \tag{12}$$

where the diffusion intensity D_J is given by:

$$D_J(J) = \langle \langle \Delta J \rangle \rangle = \sum_n n^2 |V_n|^2 \tag{13}$$

where V_n are the harmonic amplitudes in the Fourier expansion of V in the 2π periodic variable Ψ. For the correlator the phase-averaged evolution equation is:

$$\begin{aligned}\frac{\partial K}{\partial t} &= \left(\omega(J) - \omega(\bar{J}) \right) \frac{\partial K}{\partial \varphi} + \frac{\partial}{\partial J} \left(D_J(J) \frac{\partial K}{\partial J} \right) + \\ &+ \frac{\partial}{\partial \bar{J}} \left(D_J(\bar{J}) \frac{\partial K}{\partial \bar{J}} \right) + \left(D_\Psi(J) + D_\Psi(\bar{J}) \right) \frac{\partial^2 K}{\partial \varphi^2} \\ &+ F_J(J, \bar{J}, \varphi) \left(\frac{\partial \bar{f}(J)}{\partial J} \frac{\partial \bar{f}(\bar{J})}{\partial \bar{J}} + \frac{\partial^2 K}{\partial J \partial \bar{J}} \right) + F_\Psi(J, \bar{J}, \varphi) \frac{\partial^2 K}{\partial \varphi^2}\end{aligned} \tag{14}$$

where we introduced the phase difference $\varphi = \Psi - \bar{\Psi}$ and the functions D_J, D_Ψ, F_J, F_Ψ are:

$$\begin{aligned}D_J(J) &= \sum_n n^2 |V_n|^2 \\ D_\Psi &= \sum_n \left| \frac{\partial V_n}{\partial J} \right|^2 \\ F_J(J, \bar{J}, \varphi) &= \sum_n n^2 V_n(J) \bar{V}_{-n}(\bar{J}) e^{in\varphi} \\ F_\Psi(J, \bar{J}, \varphi) &= \sum_n \frac{\partial V_n(J)}{\partial J} \frac{\partial \bar{V}_{-n}(\bar{J})}{\partial \bar{J}} e^{in\varphi}\end{aligned} \tag{15}$$

IV. Asymptotic solution.

In the absence of noise, the solution of equation (14) is trivial as only the first term in the r.h.s. survives. The correlation "decay" or rather decohere due to the phase-mixing as $K(t) = \sum_m K_m(I) e^{i(m\phi + m\omega(I)t)}$ where $K_m(I)$ are the Fourier amplitudes. For nonzero but small noise $\eta \sim V^2$ the time scale of decoherence $\tau \sim 1/\lambda\sigma$ (where $\lambda = \frac{d\omega}{dI}$ and σ is the r.m.s. value of J for the distribution $\bar{f}$) is much shorter than difffusion time scale $\tau_d \sim \frac{\sigma}{\eta}$. Furthermore, the correlation "injection", that is provided by the inhomogeneous term in (14), varies only on the slow time scale. As a result, one has a quasistationary equilibrium correlation density that is the balance between the slowly changing "injection" of correlations and their fast decay.

To analyze the quasistationary solution, we drop the time derivative of K in equation (14). Another simplification comes from noticing that the correlator K is the largest at a small spatial scale $q = J - \bar{J}$, where the "decoherence" term (first term in the r.h.s. of equation (14)) is small. Expanding all the coefficients in the equation (14) to the leading order in q and keeping only the dominant derivatives in q yields:

$$\lambda q \frac{\partial K}{\partial \varphi} + 2D_J \frac{\partial^2 K}{\partial q^2} + \sum_n R_n e^{in\varphi} = 0 \qquad (16)$$

where the quantities $\lambda = \frac{d\omega(J)}{dJ}$, $D_J = D_J(J)$ and $R_n = n^2 |V_n(J)|^2 \left(\frac{\partial \bar{f}(J)}{\partial J}\right)^2$ depend on J as a parameter. For the nonzero harmonics of K in φ one obtains:

$$in\lambda q K_n + 2D_J \frac{\partial^2 K_n}{\partial q^2} + R_n = 0 \qquad (17)$$

For the power spectrum of the fluctuations $\tilde{K_n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dq\, K_n(q) e^{ikq}$ the resulting equation is of the first order:

$$n\lambda \frac{\partial \tilde{K_n}}{\partial k} - D_J k^2 \tilde{K_n} + R_n \delta(k) = 0 \qquad (18)$$

and allows an explicit solution:

$$\tilde{K_n}(k) = \begin{cases} -\frac{R_n}{n\lambda} \exp\left(\frac{D_J}{3n\lambda}k^3\right) & \text{if } nk\lambda < 0 \\ 0 & \text{otherwise} \end{cases} \qquad (19)$$

This is the central result of our analysis. The "correlation radius " q_c is $q_c \sim \left(\frac{D_J}{|3n\lambda|}\right)^{1/3}$. For small noise/large nonlinearity the "correlations radius" is small, which corresponds to the short-wavelength fluctuations ("microstructure") of the beam density.

A special feature of the spectrum (19) is its disconinuity. It is easy to see that this discontinuity is the manifestation of the long $\sim 1/q$ "tail" of the correlator K. Indeed, for large $q \ll q_c$ the second term in equation (17) becomes much smaller than the first, and one obtains $1/q$ tail. It is possible to obtain a more general expression for the "tail" $q \gg q_c$ not limited by the condition $q \ll J$ by keeping the same terms of the primary evolution equation(14) (i.e. the first term in the r.h.s. and the inhomogeneous term) without expanding in q. The resulting expression for the correlator "tail" is:

$$K_n(J, \bar{J}, t) = \frac{iV_n(J)V_n^*(\bar{J})}{n(\omega(J) - \omega(\bar{J}))} \frac{\partial \bar{f}(J,t)}{\partial J} \frac{\bar{f}(\bar{J},t)}{\partial \bar{J}} \qquad (20)$$

The most important quantity characterising the fluctuations is their intensity, which is the value of the correlator K at $q = 0$, and can be calculated by integrating the spectrum $\tilde{K_n}$. The resulting intensities $P_n = K_n(0)$ are:

$$P_n(J) = \frac{\Gamma(1/3)}{3^{2/3}} \frac{R_n(J)}{(n\lambda(J))^{2/3} D_J^{1/3}(J)} \qquad (21)$$

Thus, the fluctuations intensities are of the order $P_n \sim (\eta/\lambda)^{2/3}$ and will be small for small noise /large nonlinearity.

V. Conclusions.

We presented the evolution equation formalism for the correlation function of the density distribution fluctuations in the nonlinear oscillator under the influnce of "coherent" (same for all particles) noise. For the weak noise/ large nonlinearity of oscillations the fluctuations are small and short-ranged. The mechanism of the loss of correlations is related to the "decoherence" of oscillations due to amplitude-dependent frequency of oscillations, and since it is not a regular dissipative mechanism, the correlations demonstrate a long $\sim 1/q$ tail in the energy difference.

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