Three Dimensional Multipole Decomposition of Fields*

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Abstract

A new method to generate the general multipole representation of the three dimensional static field, electric or magnetic, is obtained via a scalar potential evaluated from the arbitrary specified source. As an application of this formulation, a previously described 3-D electric field decomposition method has been further generalized to the magnetic field.

1. INTRODUCTION

Representing an arbitrary three-dimensional vector field requires enormous amount of information. Multipole expansion is the natural and efficient way of representing a field with symmetry. A good example is the field from a quadrupole magnet which consists of a large quadrupole component with relatively small fringe fields. Then the multipole expansion converges rapidly and from the symmetry of the magnet geometry it can be easily seen that certain multipoles does not occur.

For a static field, electric or magnetic, the Green's function is well known, and the multipole coefficients can be determined from the source of the field. For a electrostatic problems the potential at the electrode is usually given and the charge density can be obtained by the capacity matrix technique[1] without solving for the field everywhere. For the magnetostatic problem, the current source is usually given.

In this report general multipole decomposition method for the static vacuum field from an arbitrary source is presented. In section II, the multipole expansion of the field is defined and the method of generating its coefficients from the Green's function is described. Section III shows the result from its application to a simple magnet geometry. A summary and conclusion is given in Sec. IV.

2. MULTIPOLE EXPANSION

Static vacuum fields, electric or magnetic, can be represented by a scalar potential. The scalar potential can be expressed in terms of multipoles which exploit the polar symmetry of the system. The convergence of the expansion depends on the system of interest, however, most of simple designs such as quadrupoles or sextupoles have a single dominant component in addition to the many small other multipole terms. Then the field can be accurately represented by keeping a few leading terms.

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The multipole coefficients $M_{k,l}(z)$ of the potential ϕ are defined in cylindrical coordinates system by

$$\phi(\rho,\theta,z) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} M_{k,l}(z)\,\rho^k\cos(l\theta) \tag{1}$$

for the system of up-down symmetry. No z-axis expansion is performed and $M_{k,l}(z)$ is calculated at numerous locations in z.

The source-free vacuum potential ϕ satisfies the Laplace equation ($\nabla^2\phi = 0$) and thus the $M_{k,l}$ observe the following recursion relation:

$$M_{k,l} = M''_{k-2,l}/(l^2 - k^2), \tag{2}$$

where double prime denotes the second derivative with respect to z. In order for ϕ be analytic near the origin, the relation $k \geq l \geq 0$ and $k - l = even$ must be true for non-zero coefficients. The entire ensemble of multipole coefficients can then be determined from $M_{l,l}$ and its $z-$derivatives.

Since the field can be determined from the Green's function which is analytic away from the source, it is possible to decompose the Green's function into multipoles and the total multipole coefficients are obtained by integration over the source.

Electric potential ϕ from the charge distribution $Q(\mathbf{x}')$ is given by (setting $4\pi\epsilon_o \rightarrow 1$)

$$\phi = \int d\mathbf{x}' G_e(\mathbf{x},\mathbf{x}')Q(\mathbf{x}') \tag{3}$$

where

$$G_e(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|}. \tag{4}$$

Away from the charges the Green's function G is infinitely differentiable, and it is in principle possible to compute the multipole coefficients $M_{k,j}$ by differentiating equation (1). Hence the multipole coefficient of the electrostatic field at the origin has the form

$$M_{k,l} = \int d\mathbf{x}\, K_{k,l}(\mathbf{x})\, Q(\mathbf{x}) \tag{5}$$

and the explicit expression of $K_{k,l}$ is given in the Table 1.

The magnetic field is determined from the current source $\mathbf{I}$ by Biot-Savart's law (setting $\mu_o/4\pi \rightarrow 1$),

$$\mathbf{B} = -\int d\mathbf{x}' \mathbf{G}_m(\mathbf{x},\mathbf{x}') \times \mathbf{I}(\mathbf{x}')$$

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where $\mathbf{G}_m$ is the Green's function for the magnetic field and its explicit form is given by

$$\mathbf{G}_m(\mathbf{x}, \mathbf{x}') = \frac{\mathbf{x} - \mathbf{x}'}{r^3}$$

where

$$r = |\mathbf{x} - \mathbf{x}'|.$$

After some algebra the multipole coefficient $M_{k,l}$ of the magnetic field at the orgin can be recast as,

$$M_{k,l} = -\int d\mathbf{x}\, [K^x_{k,l}\, I_x(\mathbf{x}) + K^y_{k,l}\, I_y(\mathbf{x}) + K^z_{k,l}\, I_z(\mathbf{x})] \quad (6)$$

with the explicit form of the vector coefficients $\mathbf{K}_{k,l}$ given in the Table 2.

The zeroth order coefficient $M_{0,0}$ is determined by the integral along the z-axis

$$M_{0,0}(\mathbf{x}) = -\int_0^{\mathbf{x}} B_z(\mathbf{x}')d\mathbf{x}'.$$

3. COMPUTATIONAL RESULTS

It is straight forward to evaluate the multipole coefficients from the expressions (5) and (6). In principle multipole coefficients of all orders, still exact, can be obtained from the method above, however, in the present work the expansion was truncated at 10^{th} order since the number of coefficients increases by the square of the order and the expansion usually converges rapidly for the typical system of interest.

The coefficients are made dimensionless by choosing the scaling length of aperture radius a and the proper scalar potential value at the aperture radius and $\theta = 0$, i.e.:

$$\phi = \phi_o \sum_{k=0}^{10} \sum_{l=0}^{10} M_{k,l}(z) \left(\frac{\rho}{a}\right)^k \cos(l\theta) \quad (7)$$

Test runs made for an electrostatic quadrupole show an exact match to the previous calculation using the Differential Algebra (DA) technique [1] except the computation time is reduced by at least two orders of magnitude. Typical computation times spent for the extraction of the coefficients up to tenth order is less than 1 second on a Cray computer (XMP) from the charge nodes of 5000 points. Calculation of the charges on the nodes which involves the inversion of the capacity matrix takes about 800 Cray cpu seconds as before. For the magnetic multipole expansion, two test runs are presented for simple geometries. Case one is the simple Helmholtz coil consisting of two identical circular rings of radius a separated by the equal distance a. Axi-symmetry of the field gives zero coefficients except the $M_{k,0}$ for even k. This is an excellent example showing advantage of the multipole expansion since only a few coefficients are needed to describe the 3-D vector field at fixed z. Figure 1 shows the non-zero multipole coefficients up 10^{th} order. Though not plotted, the axial magnetic field at the axis can be obtained from the z-derivative of $M_{0,0}$ and the slope of the $M_{0,0}$ in the plot is seen to be nearly constant as expected.

A second case is the simple magnetic quadrupole made of current elements shown in Figure 2. A single unit of current is flowing on the arc-shaped current elements of aperture radius a and two units of current are on the straight segment running along the z-axis. The length of the straight section is chosen to be the same as the aperture radius (a) so that a rich content of multipoles from the fringe field is produced. From the symmetry of the current geometry, all the multipole coefficients with l other than $2 + 4k$ for a non-negative integer k are zero. Figure 3 shows the non-zero multipole coefficients of the simple test current distribution of the quadrupole in the Figure 2.

5. CONCLUSION AND DISCUSSION

The general multipole expansion method of the static field is presented. The previous multipole calculation of the three-dimensional electrostatic field from the arbitrary electrode geometry [1] has been generalized to the magnetic field. In addition, using symbolic algebra, instead of differential algebra[2], each multipole coefficient is explicitly calculated, hence the computation time is reduced substantially (by a factor of hundred or greater), which makes this method practical.

6. REFERENCES

[1] M. Berz, W. Fawley and K. Hahn, Nucl. Inst. and Method, A 307 (1991) 1.

[2] M. Berz, IEEE Trans. Elec. Dev., 35-11 (1988) 2002.

Table 1. Electric multipole coefficient $K_{k,l}$ at origin from a unit source at $\mathbf{x}$. Here $r = \sqrt{x^2 + y^2 + z^2}$.

$$K_{0,0} = \frac{1}{r}$$

$$K_{1,1} = \frac{x}{r^3}$$

$$K_{2,2} = \frac{3\,(x^2 - y^2)}{4\,r^5}$$

$$K_{3,3} = \frac{5\,(x^3 - 3\,x\,y^2)}{8\,r^7}$$

$$K_{4,4} = \frac{35\,(x^4 - 6\,x^2\,y^2 + y^4)}{64\,r^9}$$

$$K_{5,5} = \frac{63\,(x^5 - 10\,x^3\,y^2 + 5\,x\,y^4)}{128\,r^{11}}$$

$$K_{6,6} = \frac{231\,(x^6 - 15\,x^4\,y^2 + 15\,x^2\,y^4 - y^6)}{512\,r^{13}}$$

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$$K_{2,0} = \frac{x^2 + y^2 - 2\,z^2}{4\,r^5}$$

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Table 2. Magnetic multipole coefficient $K^i_{k,l}$ at origin from a unit current source I_i at $\mathbf{x}$. Here $r = \sqrt{x^2+y^2+z^2}$.

$$K^x_{0,0} = \frac{y\,z}{(x^2+y^2)\,r}$$
$$K^x_{1,1} = K^x_{3,1} = K^x_{5,1} = K^x_{7,1} = K^x_{9,1} = 0$$
$$K^x_{2,2} = \frac{3\,y\,z}{4\,r^5}$$
$$K^x_{3,3} = \frac{5\,x\,y\,z}{4\,r^7}$$
$$K^x_{4,4} = \frac{35\,y\,(3\,x^2-y^2)\,z}{64\,r^9}$$
$$K^x_{5,5} = \frac{63\,x\,y\,(x^2-y^2)\,z}{32\,r^{11}}$$
$$K^x_{6,6} = \frac{231\,y\,(5\,x^4-10\,x^2\,y^2+y^4)\,z}{512\,r^{13}}$$
$$K^x_{2,0} = \frac{3\,y\,z}{4\,r^5}$$

$$K^y_{0,0} = -\frac{x\,z}{(x^2+y^2)\,r}$$
$$K^y_{1,1} = \frac{z}{r^3}$$
$$K^y_{2,2} = \frac{3\,x\,z}{4\,r^5}$$
$$K^y_{3,3} = \frac{5\,(x^2-y^2)\,z}{8\,r^7}$$
$$K^y_{4,4} = \frac{35\,x\,(x^2-3\,y^2)\,z}{64\,r^9}$$
$$K^y_{5,5} = \frac{63\,(x^4-6\,x^2\,y^2+y^4)\,z}{128\,r^{11}}$$
$$K^y_{6,6} = \frac{231\,x\,(x^4-10\,x^2\,y^2+5\,y^4)\,z}{512\,r^{13}}$$
$$K^y_{2,0} = -\frac{3\,x\,z}{4\,r^5}$$

$$K^z_{0,0} = K^z_{2,0} = K^z_{4,0} = K^z_{6,0} = K^z_{6,0} = K^z_{10,0} = 0$$
$$K^z_{1,1} = -\frac{y}{r^3}$$
$$K^z_{2,2} = \frac{-3\,x\,y}{2\,r^5}$$
$$K^z_{3,3} = \frac{5\,y\,(-3\,x^2+y^2)}{8\,r^7}$$
$$K^z_{4,4} = \frac{35\,x\,y\,(-x^2+y^2)}{16\,r^9}$$
$$K^z_{5,5} = \frac{63\,y\,(-5\,x^4+10\,x^2\,y^2-y^4)}{128\,r^{11}}$$
$$K^z_{6,6} = \frac{231\,x\,y\,(-3\,x^4+10\,x^2\,y^2-3\,y^4)}{256\,r^{13}}$$
$$K^z_{3,1} = \frac{3\,y\,(-x^2-y^2+4\,z^2)}{8\,r^7}$$

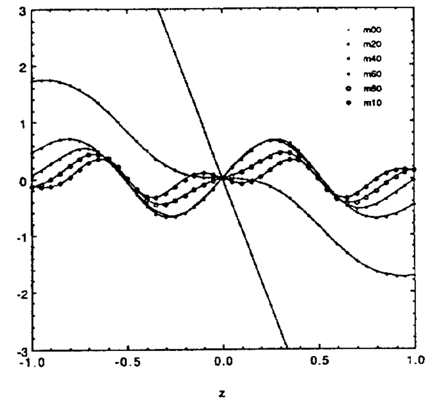

Fig. 1 - Multipole coefficients of the Helmholtz coil up to 10^{th} order. All none-axisymmetric coefficients are zero. In Helmholtz coil, axial separation of the identical rings is equal to their radius (a).

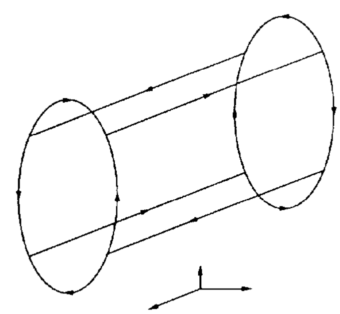

Fig. 2 - Current distribution in a simple quadrupole geometry. A single unit of current is on the arc shaped elememts and two units of current is on the straight section in order to prevent charge accumulation.

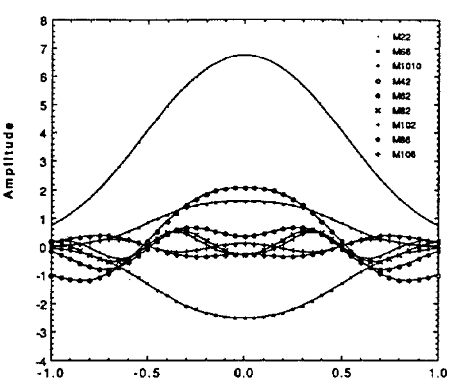

Fig. 3 - Multipole coefficients $M_{k,l}$ for the simple quadrupole geometry in Figure 2. The length of the straight section element is chosen to be the same as the aperture radius. From the symmetry, coefficients of l= 0, 1, 3, 4, 5, 7, 8, and 9 are zero.