Example Application for the Hamiltonian Description of an Azimuthally Varying Field Racetrack Microtron

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Abstract

A useful method for obtaining stable transverse motion in a (racetrack) microtron is the application of bending magnets with an azimuthally varying field (AVF) profile. A Hamiltonian theory has been set up to describe the reference orbit as well as the optical properties in both transverse directions for an AVF magnet with an arbitrary field profile. We recapitulate the main analytical results of the Hamiltonian theory and compare these to the results of numerical calculations for a relevant example AVF profile.

I. INTRODUCTION

For cyclotrons, it is well known that simultaneous horizontal and vertical orbit stability as well as isochronism can be achieved by subjecting the beam to an azimuthally varying magnetic field. We apply similar ideas to a (racetrack) microtron, i.e. we superimpose an azimuthally varying field (AVF) profile on the main average magnetic field of the bending magnets. When such magnets are designed properly, quadrupoles in the drift space and solenoids on the cavity axis are no longer needed to focus the beam.

As the modulation of the magnetic field is assumed to be small, a first order solution for the particle motion has been derived. In this paper, we will compare these analytical results with numerical calculations in order to verify the first order equations and to examine higher order effects.

II. ANALYTICAL RESULTS

In this section, we recapitulate the main analytical results, obtained in reference [1]. A schematic overview of the geometry is given in Fig. 1. We consider a bending magnet in a polar coordinate system (r, ϑ, z). The median plane is the $z = 0$ plane. A test particle is injected into the magnet at the origin of the righthanded coordinate system (r, ϑ, z). The median plane field B_z (pointing in the positive z direction) is assumed to depend only on ϑ and is split into a constant main field B_0 and a flutter profile $f(\vartheta)$

$$B_z(\vartheta) = B_0[1 + f(\vartheta)], \quad |f(\vartheta)| \ll 1, \quad 0 \le \vartheta \le \tfrac{1}{2}\pi.$$

We assume $f(0) = (df/d\vartheta)_0 = 0$. The pole edge where the beam exits the magnet is located at $\vartheta = \frac{1}{2}\pi$. Via a suitable choice of the vector potential, the magnetic field is incorporated in a relativistic Hamiltonian decribing the

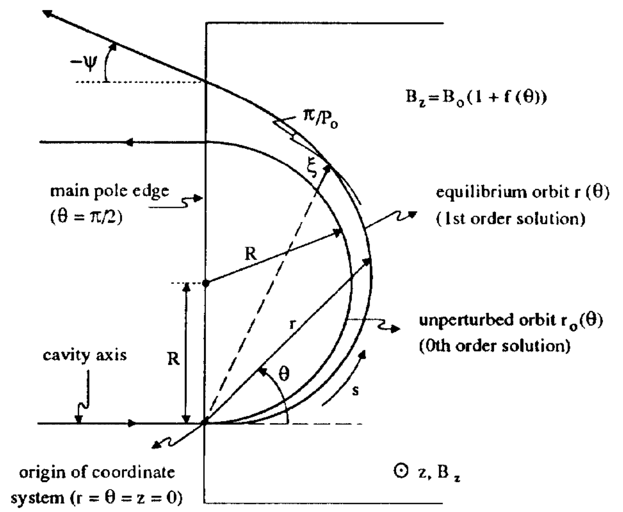

Figure 1: *Schematic overview of the considered geometry.*

median plane particle motion with ϑ as independent variable. From the solution for the equilibrium orbit up to first order we can derive expressions for the exit angle ψ (defined as the angle relative to the pole boundary normal vector) and orbit length s through the magnet. We obtain

$$\begin{aligned} \psi &= -2\int_0^{\pi/2} f(\vartheta)\cos(2\vartheta)d\vartheta, \\ s &= R\left\{\pi + \psi - 2\int_0^{\pi/2} f(\vartheta)d\vartheta\right\}, \end{aligned}$$

where R is the reference radius, defined as $R = P_0/(eB_0)$, with e the electron charge and P_0 the total kinetic momentum. The angle ψ should normally be chosen zero for the sake of closed orbits.

The linear, transverse oscillations with respect to the equilibrium orbit, either horizontally (x) or vertically (z), are derived from Hamiltonians and can be expressed as phase space transfer matrices M_x and M_z

$$\begin{pmatrix} y \\ y' \end{pmatrix}_\vartheta = M_y(\vartheta)\begin{pmatrix} y_0 \\ y_0' \end{pmatrix}, \quad y \in \{z, x\}, \quad y' = dy/ds.$$

In the present paper, we only consider the trace of the transfer matrices as a function of azimuth. These read

$$\begin{aligned} \mathrm{Tr}^z(\vartheta) &= 2 + [a - \tfrac{1}{2}(dF/d\vartheta)]\vartheta, \\ \mathrm{Tr}^x(\vartheta) &= 2\cos(2\vartheta) - 2[G + \tfrac{1}{8}(dF/d\vartheta)]\sin(2\vartheta), \end{aligned}$$

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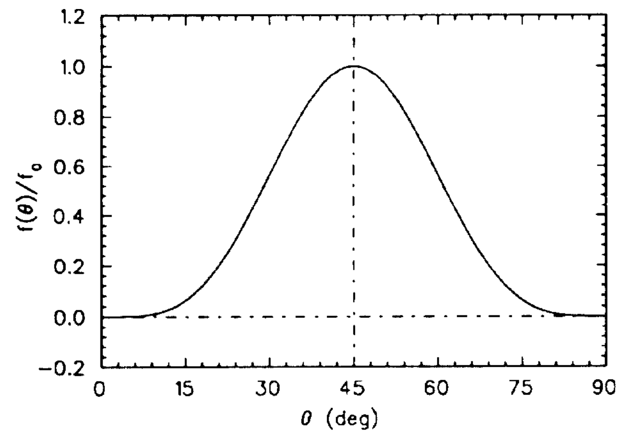

Figure 2: *Shape of the applied AVF modulation.*

with

$$F(\vartheta) = \frac{1}{\sin^2(\vartheta)} \int_0^\vartheta f(t)\sin(2t)dt, \quad G(\vartheta) = \int_0^\vartheta g(t)dt,$$

$$g(\vartheta) = 2f - 2F - \frac{1}{2\tan(\vartheta)}\frac{df}{d\vartheta} - \frac{1}{8}\frac{d^2F}{d\vartheta^2},$$

$$a(\vartheta) = -\int_0^\vartheta \left[\frac{2}{\tan(\vartheta)}\frac{df}{d\vartheta} - \frac{1}{2}\frac{d^2F}{d\vartheta^2}\right] dt.$$

By evaluating the above expressions at $\vartheta = \pi/2$, we can find the matrix traces $\mathrm{Tr}^y_{\pi/2}$ for half the revolution

$$\mathrm{Tr}^z_{\pi/2} = 2 + \tfrac{1}{2}\pi\bar{a}, \quad \mathrm{Tr}^x_{\pi/2} = -2,$$

with

$$\bar{a} \equiv a(\pi/2) = -\int_0^{\pi/2} \frac{2f(\vartheta)}{\sin^2(\vartheta)} d\vartheta.$$

Using the mirror symmetry of the equilibrium orbit (assuming $\psi = 0$), we can also derive the traces Tr^y_π for a full revolution through a 'classical' microtron (no drift space). We obtain in first order

$$\mathrm{Tr}^z_\pi = 2 + 2\pi\bar{a}, \quad \mathrm{Tr}^x_\pi = 2.$$

III. NUMERICAL CALCULATIONS

In order to check the above analytical first order results, exact numerical calculations have been done for various profiles. In this paper, we consider one specific profile and examine the effect of its amplitude on exit angle, orbit length and focusing properties in both transverse planes. The profile we consider is

$$f(\vartheta) = f_0 \sin^4(2\vartheta),$$

being a smooth hill ($f_0 > 0$) or valley ($f_0 < 0$), centered around $\vartheta = \pi/4$, see Fig. 2. For this specific profile, we obtain with our first order theory

$$\psi = 0, \quad s = \pi R(1 - 3f_0/8), \quad \bar{a} = -\pi f_0,$$
$$\mathrm{Tr}^z_{\pi/2} = 2 - \tfrac{1}{2}\pi^2 f_0, \quad \mathrm{Tr}^z_\pi = 2 - 2\pi^2 f_0.$$

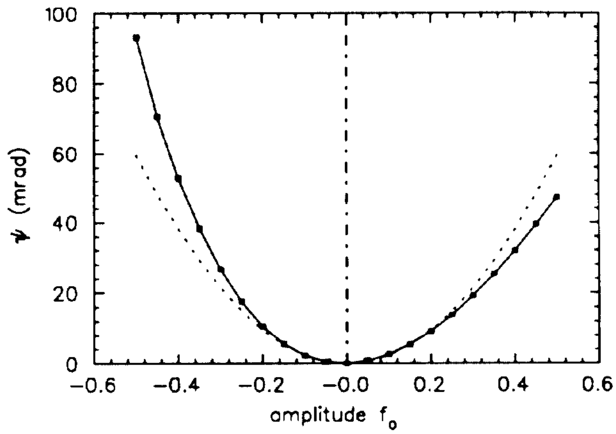

Figure 3: *Exit angle ψ as a function of f_0.*

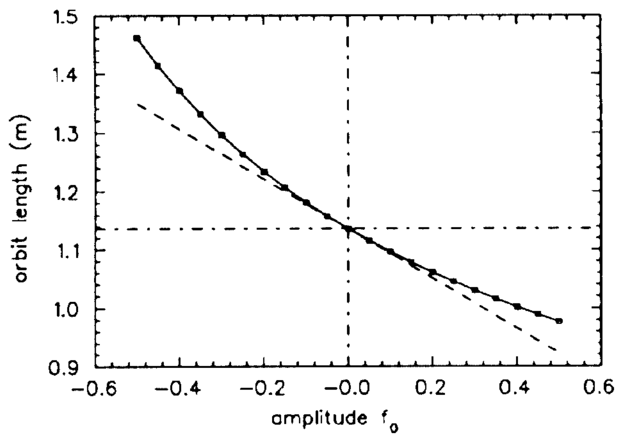

Figure 4: *Orbit length s as a function of f_0.*

For the numerical calculations we consider the interval $-0.5 < f_0 < 0.5$ as to get a good view on effects higher order in f_0. All calculations were done for $P_0 = 20$ MeV/c electrons and $B_0 = 0.19$ T.

Fig. 3 shows the exit angle ψ as a function of f_0 as obtained from numerical calculations. The curve has been fitted with a fifth order polynomial in f_0. It turns out that there is no first order term, fully in accordance with our first-order result $\psi = 0$. The dotted curve represents only the second order term of the polynomial. From this we infer that a second order theory could give a much more accurate expression for the exit angle, hence also a more accurate condition for keeping the orbits closed.

In Fig. 4, the total orbit length is plotted against f_0. The dashed, sloping line represents our first order result. It convincibly touches the numerical curve in $f_0 = 0$. The difference between both curves increases with increasing $|f_0|$, but once again, we see that this difference could be highly reduced by a second order description, as required for the sake of the isochronism condition.

The linear transverse motion in both transverse directions was numerically calculated as a function of azimuth. From the resulting matrices, the trace as a function of azimuth was extracted, its zero order part removed and the

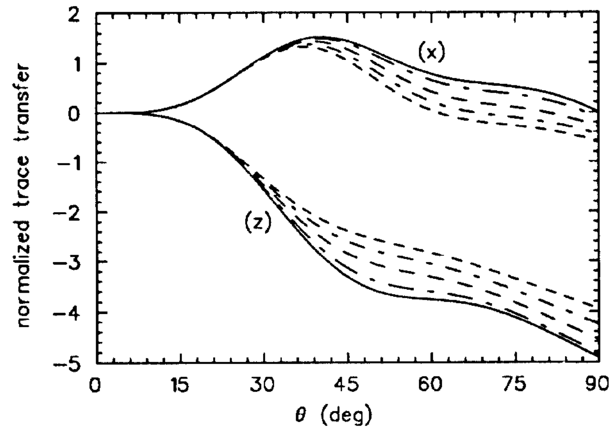

Figure 5: *Normalized horizontal and vertical trace as a function of azimuth ϑ through a single magnet.*

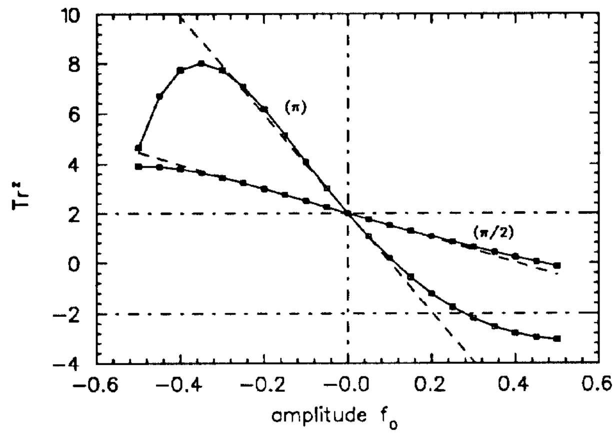

Figure 6: *Vertical traces Tr^z_π, $Tr^z_{\pi/2}$ as a function of f_0.*

remaining part divided by the amplitude f_0. The resulting curve (the 'normalized trace') is in first order independent of f_0, hence any f_0 dependency represents higher order terms. The drawn lines in Fig. 5 are the normalized horizontal and vertical traces as a function of azimuth as derived from the first order theory. The dashed lines are the result of the numerical calculations for the cases $f_0 = 0.1$, 0.3, 0.5 and 0.7. The higher order deviation gets larger with increasing f_0, but the overall shape of the curves is retained and the values at $\vartheta = \pi/2$ are still very close to the first order result.

In Fig. 6, the matrix trace for vertical motion as a function of amplitude is shown. The two drawn curves represent the numerical results for half an orbit through a microtron (labeled $\pi/2$) as well as for a full orbit (π). The dashed lines represent our first order analytical results. The agreement for half the orbit is excellent over the entire amplitude range. For the full orbit, higher order effects become significant for amplitudes larger than 0.3. It is interesting to note that the numerically obtained curve for Tr^z_π bends back to the stable region for $f_0 < -0.3$; this means that for $f_0 \approx -0.55$, vertical motion can be stable again, but it is governed by higher order effects in f_0 and

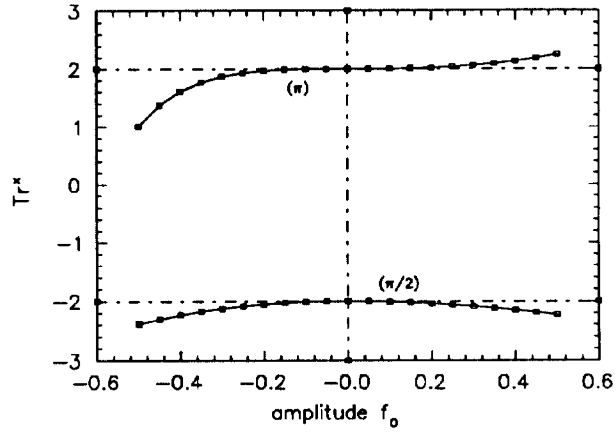

Figure 7: *Horizontal traces Tr^x as a function of f_0.*

therefore very sensitive to small changes in f_0.

In Fig. 7, the matrix traces for horizontal motion as a function of amplitude are drawn. For half the orbit, we see that there is a weak second order contribution that moves the trace value outside the stable region for either sign of f_0. For the full orbit, it was predicted by the first order theory that there could be no second order effect of f_0 on the trace [1]. Indeed, we see that the curve of Tr^x_π is antisymmetric around f_0. A least squares fit of the results with a fifth order polynomial in f_0 proved that no second order term is present in the curve.

Combining the results of Fig. 6 and 7, we see that, for the present profile, simultaneous horizontal and vertical beam stability in a 'classical' microtron is not possible with small values of f_0. This same conclusion has been derived in general for the case of a racetrack microtron (with driftspace) in reference [1]. As a solution, we rotate the bending magnets of the racetrack microtron through the median plane (but keeping the orbits closed), thus introducing additional quadrupole effects at the magnet entrance and exit. For a classical microtron (no drift space) this solution cannot be used.

IV. CONCLUSIONS

We have compared the analytical results of our first order description of the azimuthally varying field (racetrack) microtron with numerical results for one specific AVF profile. The analytical theory shows excellent agreement with the numerical calculations up to first order. Second order effects could be important for determining the exit angle and orbit length, but focusing properties are sufficiently accurate in first order for flutter profile amplitudes up to 30%.

V. REFERENCES

[1] J.L. Delhez and W.J.G.M. Kleeven, "Canonical Treatment of an Azimuthally Varying Field Racetrack Microtron," submitted to *Particle Accelerators*.