Saturation of a High Gain FEL*

R.L. Gluckstern†, S. Krinsky‡ and H. Okamoto§

Abstract

We study the saturated state of an untapered free electron laser in the Compton regime, arising after exponential amplification of an initial low level of radiation by an initially monoenergetic, unbunched electron beam. The saturated state of the FEL is described by oscillations about an equililbrium state. Using the two invariants of the motion, and certain assumptions motivated by computer simulations, we provide approximate analytic descriptions of the radiation field and electron distribution in the saturation regime. We first consider a one-dimensional approximation, and later extend our approach to treat an electron beam of finite radial extent. Of note is a result on the radiated power in the case of an electron beam with small radius.

I. INTRODUCTION

In this paper we study the saturated state of an untapered FEL in the Compton regime. Guided by the results of simulations starting with a monoenergetic unbunched electron beam and a low initial level of radiation, we make assumptions which prove to give an accurate picture of what happens in the saturation regime. The solutions in the saturation regime are related to the initial conditions by using the two invariants of the motion. Finally we extend our one-dimensional model to treat an electron beam of finite radial extent, including the effects of the diffraction of the radiation and the radiation focussing properties of the electron beam bunched by the FEL interaction. This work will be presented in greater detail[1].

The starting point of the analysis is the scaled equations for the evolution of the one dimensional electron distribution and for the monochromatic radiation field. The notation is that of Bonifacio et. al.[2] and the equations are

$$\frac{d\sigma_j}{d\tau} = p_j, \tag{1}$$

*Work supported by the Department of Energy

†University of Maryland, College Park, MD 20742
‡Brookhaven National Laboratory, Upton, NY 11973
§University of Maryland, College Park, MD 20742; on leave from: Institute for Chemical Research, Kyoto University, Japan

$$\frac{dp_j}{d\tau} = -Ae^{i\sigma_j} - A^* e^{-i\sigma_j}, \tag{2}$$

$$\frac{dA}{d\tau} = \langle e^{-i\sigma_j}\rangle + iA\delta, \tag{3}$$

where σ_j and p_j are the phase of the j^{th} electron relative to the radiation and its (scaled) momentum deviation, A is the (scaled) radiation amplitude at the (scaled) longitudinal position $\tau = 2\rho k_w z$, where $2\pi/k_w$ is the wiggler period and ρ is the Pierce parameter, δ is the detuning of the laser, and $\langle\rangle$ is an average over the electron distribution.

It is easy to show from Eqs. (1)-(3) that

$$\langle p_j\rangle + |A|^2 = C_1 \tag{4}$$

and

$$\frac{\langle p_j^2\rangle}{2} + 2Im[A\langle e^{i\sigma_j}\rangle] - \delta|A|^2 = C_2 \tag{5}$$

are constants of the motion. For an initially monoenergetic unbunched electron beam and a low initial level of radiation, the constants C_1, C_2 are taken to be zero.

II. EQUILIBRIUM DISTRIBUTION

In Fig. 1 we show a typical evolution of the radiation with τ. The field builds up exponentially as the electrons bunch. After the bunched electrons are captured in buckets, the radiation oscillates with modest amplitude about an equilibrium distribution. The approximations in our model are to consider only up to linear terms in the amplitude of these oscillations, and to consider only the lowest harmonic frequency of these oscillations.

In Fig. 2 we show the phase of the radiation, which appears to be very nearly linear with τ. We therefore write

$$A = (P + iQ)e^{i\nu(\tau-\tau_0)} \tag{6}$$

and introduce the equilibrium displaced electron phase $\phi_j(\tau)$

$$\phi_j(\tau) \equiv \sigma_j + \nu(\tau - \tau_0) + \pi/2, \tag{7}$$

requiring ν to be chosen such that $\langle\phi_j'\rangle = 0$, where the prime stands for $d/d\tau$. For zero detuning $\delta = 0$, we find in the saturation regime that all quantities oscillate about an equilibrium state for which

$$P = P_0\ ,\ Q = 0\ ,\ \nu = P_0^2, \tag{8}$$

0-7803-1203-1/93$03.00 © 1993 IEEE

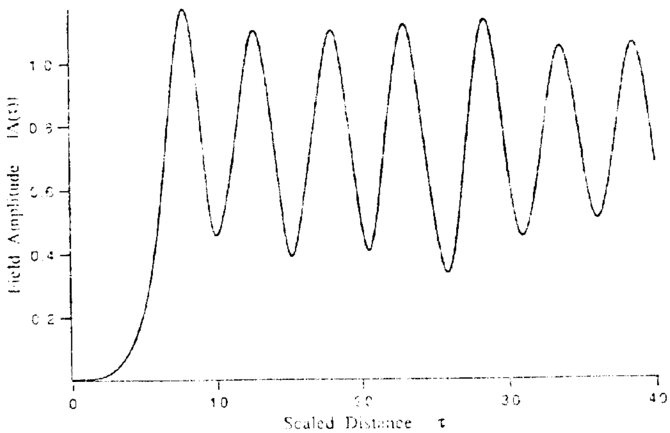

Figure 1: Evolution of radiation field amplitude $|A|$ with τ.

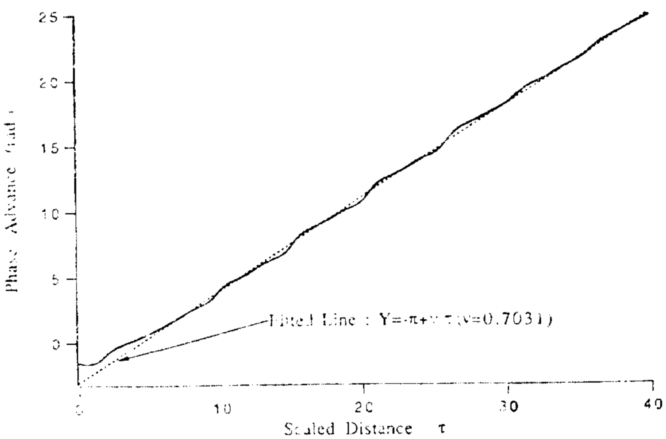

Figure 2: Phase of the radiation field as a function of τ.

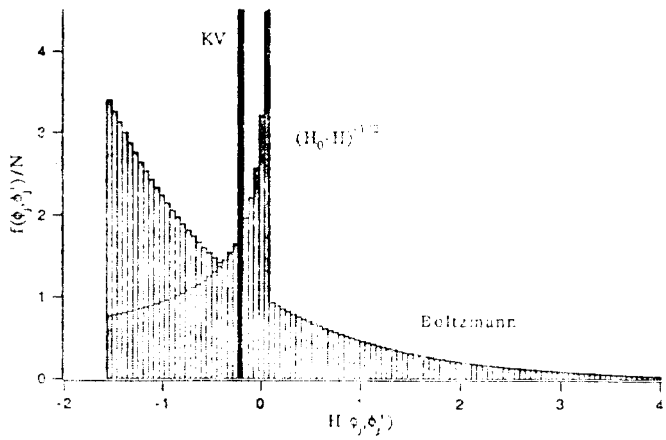

Figure 3: The three distributions, KV, $(H_0 - H)^{1/2}$ and Boltzmann plotted as functions of H defined in Eq. (10).

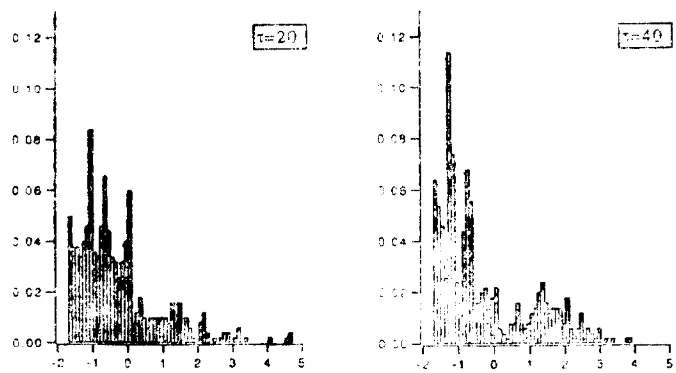

Figure 4: Electron distributions obtained from simulation for $\tau = 20, 40$ plotted as functions of H.

$$\langle \cos\phi_j \rangle = P_0^3 \ , \ \langle \sin\phi_j \rangle = 0 \ , \ \langle \phi_j'^2 \rangle = 3P_0^4. \qquad (9)$$

Equilibrium distributions satisfying the conditions in Eqs. (8) and (9) can be constructed as $f(\phi, \phi') = F(H)$, where

$$H = \phi'^2/2 - 2P_0 \cos\phi. \qquad (10)$$

We have considered three widely different distributions

$$f_{\mathrm{KV}} = N_{\mathrm{KV}} \delta(H - H_0) \qquad \text{(KV)[3]} \qquad (11)$$

$$f_{-1/2} = N_{-1/2}(H - H_0)^{-1/2} \qquad (12)$$

$$f_B = N_B \exp(-\alpha H) \qquad \text{(Boltzmann)} \qquad (13)$$

and find in all cases that $P_0 = 0.81$, in good agreement with Fig. 1. In Fig. 3 we show the three different distributions plotted as a function of H. And in Fig. 4 we show the electron distributions obtained from the simulations for $\tau = 20, 40$. The background from the electrons which are not trapped is seen to be more or less independent of H, and the distributions of the trapped electrons seems to most resemble the Boltzmann distribution.

III. EXPONENTIAL GROWTH REGIME

If we take two derivatives of Eq. (3) and consider only those terms linear in p_j and A, we find, for $\delta = 0$

$$\frac{d^3 A}{d\tau^3} = iA. \qquad (14)$$

The exponential growth regime then corresponds to the solution

$$A(\tau) \simeq A_0 \exp[(\sqrt{3} + i)\tau/2]. \qquad (15)$$

When $|A(\tau)|$ is of order 1, non-linear terms in A, p_j must be included, and some sort of saturation will take place.

IV. SATURATION REGIME

The saturated state of the FEL is described by oscillations about an equilibrium state[4, 5, 6]. This equilibrium state corresponds to a steady state solution of Eqs. (1)-(3). The proper choice of the equilibrium solution is significantly restricted[4] by the two invariants of Eqs. (4) and (5), relating properties of the saturated state back to the initial conditions at the start-up of the FEL.

We now consider oscillations about the equilibrium distribution, defining the displaced electron phase as

$$\beta_j(\tau) = \sigma_j(\tau) + \nu(\tau - \tau_0) + \pi/2. \tag{16}$$

Using Eqs. (6) and (16), we now write, with $\delta = 0$,

$$\beta_j''(\tau) = -2P\sin\beta_j - 2Q\cos\beta_j \tag{17}$$

$$Q' + \nu P = \langle\cos\beta_j\rangle \tag{18}$$

$$P' - \nu Q = \langle\sin\beta_j\rangle \tag{19}$$

together with the two invariants

$$\langle\beta_j'\rangle + P^2 + Q^2 = \nu, \tag{20}$$

$$\langle\beta_j'^2\rangle - 2\nu\langle\beta_j'\rangle + \nu^2 = 4P\langle\cos\beta_j\rangle - 4Q\langle\sin\beta_j\rangle. \tag{21}$$

We now consider oscillations about the equilibrium distribution of the form

$$P(\tau) = P_0 + P_1\cos\Omega\tau\ ,\ Q(\tau) = Q_1\sin\Omega\tau \tag{22}$$

$$\beta_j(\tau) = \phi_j(\tau) + a\sin\Omega\tau, \tag{23}$$

where the oscillation of the electrons is assumed to be coherent. Keeping only terms linear in P_1, Q_1 and a, we can show[1] that $\Omega = \sqrt{3}P_0^2 = 1.14$, slightly smaller than the value $\Omega = 1.25$ seen in the simulation in Fig. 1.

V. TRANSITION FROM THE EXPONENTIAL TO THE SATURATION REGIME

A plot of $dP/d\tau$ vs. $P(\tau)$ from the simulation shows a straight line starting at $(0,0)$, corresponding to the exponential regime, approximately tangent to a repeated elliptical orbit centered at $(0.8, 0)$, corresponding to the oscillation in the saturation regime. Postulating this model of approximate tangency for the transition from the exponential to the saturation regime leads to the prediction of $P_1 \simeq 0.49, Q_1 \simeq 0.28$, somewhat larger than the values $P_1 \simeq .40, Q_1 \simeq .20$ seen in the simulations. Considering the crude nature of the transition model, this agreement is quite good.

VI. ELECTRON BEAM WITH FINITE RADIAL EXTENT

We now extend the single harmonic model considered above to the two-dimensional case of an electron beam with finite radial extent. We ignore betatron oscillations, assuming the electron beam has no angular spread, but include the diffraction of the radiation and the radiation focusing properties of the electron beam bunched by the FEL interaction.

The equations for the electron motion are still those in Eqs. (1) and (2). But Eq. (3) for the evolution of the radiation is now changed to

$$A' - i\nabla^2 A = u(r)\langle e^{-i\sigma_j}\rangle \tag{24}$$

where $u(r)$ is the fixed electron beam density profile. We also take $\delta = 0$. The form of the two invarients is also changed somewhat. The scaled transverse coordinate is $\boldsymbol{r} = \sqrt{4\rho k_w k_s}\boldsymbol{r}_d$ where k_s is the resonant radiation wave number and $\boldsymbol{r}_d$ is the unscaled transverse coordinate vector.

The equilibrium state is now governed by the solution of the differential equation

$$\nu P_0 - \nabla^2 P_0(r) = u(r)\langle\cos\phi_j\rangle\ ,\ \langle\sin\phi_j\rangle = 0 \tag{25}$$

and the modified invariants lead to

$$\nu = \frac{\int_0^\infty r dr P_0^2(r)}{\int_0^\infty r dr u(r)},$$

$$\nu^2 = \frac{\int_0^\infty r dr u(r)\langle\phi_j'^2 - 2P_0\cos\phi_j\rangle}{\int_0^\infty r dr u(r)}. \tag{26}$$

Explicit relations can now be obtained for these parameters with the specific phase space distributions $f_a(H), b_b(H), f_c(H)$, and for a given beam profile $u(r)$.

As a result of this analysis, we obtain an equilibrium guided solution and oscillations about this solution. There are two types of oscillation modes, one guided and one corrresponding to radiation propagating to $r = \infty$. The escape of the radition from the electron beam leads to a damping of the oscillations. Also, explicit results have been obtained[1] in the limits of large and small electron-beam radius. In particular, we find that, for small beam radius, the radiated power is proportion to $I_0^{3/2}$ where I_0 is the current. This result is intermediate between the incoherent (I_0) and fully coherent (I_0^2) limits.

References

[1] R.L. Gluckstern, S. Krinsky and H. Okamoto, to be published in Phys. Rev. E, June 1993.

[2] B. Lane and R.C. Davidson, Phys. Rev. A **27**, 2008 (1983).

[3] Kapchinsky and Vladimirskiy first introduced the δ-function distribution in H for use in two-dimensional transverse beam dynamics with space charge. See, for example, I.M. Kapchinskiy, *Theory of Resonance Linear Accelerators* (Harwood Academic, New York, 1985), p. 247ff (translated from the Russian).

[4] B. Lane and R.C. Davidson, Phys. Rev. A **27**, 2008 (1983).

[5] R.C. Davidson and J.S. Wurtele, Phys. Fluids **30**, 557 (1987).

[6] W.M. Sharp and S.S. Yu, Nucl. Instrum. Meth. A **272**, 397 (1988); Phys. Fluids B **2**, 581 (1990).