# EIGENFUNCTIONS OF THE TRANSFER MATRIX IN THE PRESENCE OF LINEAR COUPLING* 

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## Abstract

This paper presents an approach to computing the change in the linear orbit parameters, due to a perturbing field that couples the $x$ and $y$ motions, by computing the change in the eigenfunctions of the transfer matrix and then using the relationship between the eigenfunctions and the orbit parameters. This can be compared with the approach [1] that computes the change in the transfer matrix and uses the relationship between the elements of the transfer matrix and the orbit parameters. For the case of coupled motion, the eigenfunction approach appears to be easier to apply than the transfer matrix approach, partly because the relationship between the transfer matrix and the linear orbit parameters is considerably more complicated in this case. Results are found for the change in the four eigenfunctions of the transfer matrix in the presence of linear coupling, and for the relationship between the eigenfunctions and the orbit parameters.

## I. THE EIGENFUNCTIONS AND THE LINEAR ORBIT PARAMETERS

The eigenfunctions may be defined in terms of the transfer matrix, $T\left(s, s_{0}\right)$,

$$
\begin{equation*}
x(s)=T\left(s, s_{0}\right) x\left(s_{0}\right) \tag{1a}
\end{equation*}
$$

In Eq. (1) $T\left(s, s_{0}\right)$ is a $4 \times 4$ matrix, $x(s)$ is a $4 \times 1$ column vector

$$
\begin{equation*}
x=x, p_{x}, y, p_{y} \tag{1b}
\end{equation*}
$$

In the absence of solenoids, $p_{x}=x^{\prime}$ and $p_{y}=y^{\prime}$. The eigenfunctions are those $x(s)$ that satisfy

$$
\begin{equation*}
T(s+L, s) x=\lambda x \tag{1c}
\end{equation*}
$$

where $L$ is the period of the magnetic guide field.
It can be shown [1] that there are 4 eigenfunctions $x_{i}(s), i=1,2,3,4$ with eigenvalues $\lambda_{i}$, and which occur in pairs such that for stable motion,

$$
x_{2}=x_{1}^{*}, \quad x_{4}=x_{4}^{*}, \quad \lambda_{2}=\lambda_{1}^{*}, \quad \lambda_{4}=\lambda_{3}^{*}
$$

It can be shown [2] that the eigenfunctions are solutions of the equations of motions, and that

$$
\begin{array}{ll}
x_{1}(s)=\exp \left(i 2 \pi \nu_{1} s / L\right) & f_{1}(s) \\
x_{3}(s)=\exp \left(i 2 \pi \nu_{2} s / L\right) & f_{3}(s) \tag{2}
\end{array}
$$

$f_{1}(s), f_{3}(s)$ are periodic in $s$ with period $L$, and $\nu_{1}, \nu_{2}$ are the normal mode tunes. Note $x_{i}$ and $f_{i}$ are both $4 \times 1$ column vectors.

[^0]
## A. The Transfer Matrix in Terms of the Eigenfunctions

Given the four eigenfunctions $x_{i}, i=1,4$ which are normalized such that $\tilde{x}_{i}^{*} S x_{i}=2 i$,
where $S$ is

$$
S=\left[\begin{array}{cccc}
0 & 1 & 0 & 0  \tag{3}\\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{array}\right]
$$

then it will be shown that one can find the transfer matrix $T\left(s, s_{0}\right)$ from

$$
\begin{align*}
T\left(s, s_{0}\right) & =(-1 / 2 i) U(s) \bar{U}\left(s_{0}\right) \\
\bar{U} & =\tilde{S} \tilde{U} S  \tag{4}\\
U & =\left[x_{1} x_{2} x_{3} x_{4}\right]
\end{align*}
$$

$U$ is a $4 \times 4$ matrix and $x_{i}$ is a $4 \times 1$ column vector.
Eq. (4) will be derived for the 2 -dimensional case. The generalization to 4 or more dimensions is clear. In two dimensions a solution of the equation of motion can be written as

$$
\begin{align*}
& { }^{2}=a_{1} x_{1}+a_{2} x_{2}, \quad r_{2}=x^{*}, \\
& a_{1}=\tilde{x}_{1}^{*} S x / 2 i, a_{2}=a_{1}^{*}=\tilde{x}_{2}^{*} S x /(-2 i) \tag{5}
\end{align*}
$$

Evaluate $a_{1}$ and $a_{2}$ using $x\left(s_{0}\right)$. Then

$$
\begin{align*}
& x=(1 / 2 i)\left(x_{1}(s) \tilde{x}_{1}^{*}\left(s_{0}\right)-x_{2}(s) \tilde{x}_{2}^{*}\left(s_{0}\right)\right) S x\left(s_{0}\right) \\
& x=(1 / 2 i)\left(x_{1}(s) \tilde{x}_{1}^{*}\left(s_{0}\right)-x_{1}^{*}(s) \tilde{x}_{1}\left(s_{0}\right)\right) S x\left(s_{0}\right) \\
& x=(1 / 2 i)\left[x_{1}(s) x_{1}^{*}(s)\right] S\left[\begin{array}{l}
\tilde{x}_{1}\left(s_{0}\right) \\
\tilde{x}_{1}^{*}\left(s_{0}\right)
\end{array}\right] S x\left(s_{0}\right)  \tag{6}\\
& x=(-1 / 2 i) U(s) \bar{U}\left(s_{0}\right) x\left(s_{0}\right) .
\end{align*}
$$

Thus

$$
\begin{equation*}
T\left(s, s_{0}\right)=(-1 / 2 i) U(s) \bar{U}\left(s_{0}\right) \tag{7}
\end{equation*}
$$

One may note that $V=(-2 i)^{-\frac{1}{2}} U(s)$ is symplectic as $T(s, s)=I$ and $V \bar{V}=I$.

Eq. (7) shows that knowing the eigenfunctions $x_{i}$ is equivalent to knowing the transfer matrix $T\left(s, s_{0}\right)$. Eq. (7) also shows that $T\left(s, s_{0}\right)$ is symplectic as it is the product of two symplectic matrices, $V(s)$ and $\bar{V}\left(s_{0}\right)$.
B. Relationship Between Eigenfunctions and the Linear Orbit Parameters

The eigenfunctions will now be related to the 10 linear orbit parameters for the coupled motion.

In two dimensions, the eigenfunction is related to the 3 orbit parameters $\beta, \alpha, \psi$ by

$$
\begin{equation*}
x_{1}=\binom{\beta^{\frac{1}{2}}}{\beta^{-\frac{1}{2}}(-\alpha+i)} \exp (i \psi) \tag{8}
\end{equation*}
$$

and $x_{2}=x_{1}^{*}, x_{1}$ obeys the normalization condition

$$
\begin{equation*}
\tilde{x}_{1}^{*} S x_{1}=2 i \tag{9}
\end{equation*}
$$

In four dimensions, one can go from the coordinates, $x, p_{x}, y, p_{y}$ to an uncoupled set of coordinates $v, p_{v}, u, p_{u}$ the normal coordinates, by the transformation [3]

$$
R=\left(\begin{array}{cc}
x=R & \\
I \cos \varphi & \bar{D} \sin \varphi  \tag{11}\\
-D \sin \varphi & I \cos \varphi
\end{array}\right) .
$$

$I$ and $R$ are $2 \times 2$ matrices. $I$ is the $2 \times 2$ identity matrix. $\bar{D}=D^{-1}$ and $|D|=1 . R$ is a symplectic matrix,

$$
\begin{align*}
\bar{R} R & =I \\
\bar{R} & =\tilde{S} \tilde{R} S \tag{12}
\end{align*}
$$

$\varphi$ and the 3 independent elements of $D$ may be considered as 4 of the orbit parameters. They are periodic in $s$. The other 6 orbit parameters are the $\beta_{1}, \alpha_{1}, \psi_{1}$ and $\beta_{2,} \alpha_{2}, \psi_{2}$ of the 2 normal modes.

It can be shown that $\tilde{x}^{*} S x$ is a constant [1] of the motion. Also if $x$ and $v$ are related by a symplectic matrix then

$$
\begin{equation*}
\tilde{x}^{*} S x=\tilde{v}^{*} S v \tag{13}
\end{equation*}
$$

The transfer matrix for the $v$ coordinates is given by

$$
\begin{align*}
v(s) & =U\left(s, s_{0}\right) v\left(s_{0}\right) \\
U & =\bar{R}(s) T R\left(s_{0}\right) \tag{14}
\end{align*}
$$

It can then be shown that the eigenfunction of $U, v_{i}$, and the eigenfunctions of $T$ are related by

$$
\begin{equation*}
x_{i}=R v_{i} \tag{15}
\end{equation*}
$$

The $v$ coordinates are uncoupled, so the $v_{i}$ eigenfunctions can be written down using Eq. (8) as

$$
\begin{align*}
& v_{1}=\binom{\mu_{1}}{0} \quad v_{3}=\binom{0}{\mu_{2}} \\
& \mu_{1}=\binom{\beta_{1}^{\frac{1}{2}}}{\beta_{1}^{-\frac{1}{2}}\left(-\alpha_{1}+i\right)} \exp \left(i \psi_{1}\right)  \tag{16}\\
& \mu_{2}=\binom{\beta_{2}^{\frac{1}{2}}}{\beta_{2}^{-\frac{1}{2}}\left(\alpha_{2}+i\right)} \exp \left(i \psi_{2}\right) \\
& v_{2}=v_{1}^{*}, \quad v_{4}=v_{3}^{*}
\end{align*}
$$

one may note that $\tilde{v}_{1}^{*} S v_{1}=\tilde{v}_{3} S v_{3}=2 i$. The $x_{i}$ can then be written down using $x=R V$ as

$$
\begin{align*}
& x_{1}=\binom{\mu_{1} \cos \varphi}{-D \mu_{1} \sin \varphi}  \tag{17}\\
& x_{3}=\binom{\bar{D} \mu_{2} \sin \varphi}{\mu_{2} \cos \varphi}
\end{align*}
$$

Eq. (17) relates the eigenfunctions $x_{i}$ to the 10 orbit parameters. Also $\tilde{x}_{1}^{*} S x_{1}=\tilde{x}_{3}^{*} S x_{3}=2 i$.

If the eigenfunctions are known, then Eqs. (17) can be inverted to find the 10 orbit parameters. One can use the additional relationships

$$
\begin{equation*}
\frac{d \psi_{1}}{d s}=\frac{1}{\beta_{1}}, \quad \frac{d \psi_{2}}{d s}=\frac{1}{\beta_{2}} \tag{18}
\end{equation*}
$$

$\alpha_{1}=-\frac{1}{2} \beta_{1}^{\prime}+\beta_{1} \tan \varphi d \varphi / d s, \quad \alpha_{2}=-\frac{1}{2} \beta_{2}^{\prime}+\beta_{2} \tan \varphi d \varphi / d s$ which are valid in absence of solenoidal fields.

To further illustrate how expressions for the eigenfunctions can be used to compute the linear obit parameters, consider the problem of finding $\beta_{1}$, the beta function of the normal mode with $\nu_{1}$, assuming that the eigenfunctions are known (see section II).

The eigenfunction $x_{1}$ may be written as

$$
x_{1}=\left[\begin{array}{l}
x_{11}  \tag{19}\\
x_{12} \\
x_{13} \\
x_{14}
\end{array}\right]
$$

From Eq. (17)

$$
\begin{equation*}
x_{11}=\beta_{1}^{\frac{1}{2}} \cos \varphi \exp \left(i \psi_{1}\right) \tag{20}
\end{equation*}
$$

which gives the two relations .

$$
\begin{equation*}
\psi_{1}=p h\left(x_{11}\right), \beta_{1}^{\frac{1}{2}} \cos \varphi=\left|x_{11}\right| \tag{21}
\end{equation*}
$$

where $p h\left(x_{11}\right)$ means the phase of $x_{11}$.
Assuming $x_{11}$ is known, $\psi_{1}$ can be found from Eq. (21) and one can find $\beta_{1}$ from $\beta_{1}=d \psi_{1} / d s$. Once $\beta_{1}$ is known, one can find $\cos \varphi$ from Eq. (21). One may note that the results for the tune $\nu_{1}$ and $\nu_{2}$ comes directly out of solving the equations of motion for the eigenfunctions (see section II).

To find the emittance $\epsilon_{1}$ from the eigenfunctions, one can use the relationship [4]

$$
\begin{equation*}
\epsilon_{1}=\left|\tilde{x}_{1}^{*} S x_{1}\right|^{2} \tag{22}
\end{equation*}
$$

## II. PERTURBATION SOLUTIONS FOR THE EIGENFUNCTIONS

The skew quadrupole field is described by $a_{1}(s)$. On the median plane, the field $B_{x}$ is given by

$$
\begin{equation*}
B_{x}=-B_{0} a_{1} x \tag{23}
\end{equation*}
$$

where $B_{0}$ is the main dipole field. $\rho$ is the radius of curvature of the main dipole.

The solutions of Eq. (23) were found in two previous papers $[5,6]$ when $\nu_{x}, \nu_{y}$ are near the resonance line $\nu_{x}=\nu_{y}+p$. These solutions may be written as

$$
\begin{gather*}
\eta_{x}=A \exp \left(i \nu_{x, s} \theta_{x}\right)\left\{1+\sum_{n \neq-p} f_{n}\right\}  \tag{24}\\
f_{n}=\frac{\left(\nu_{x, s}-\nu_{x}\right)}{\Delta \nu} \frac{2 \nu_{x} b_{n} \exp \left[-i(n+p) \theta_{x}\right]}{\left(n-\nu_{x}-\nu_{y}\right)(n+p)} \\
\eta_{y}=B \exp \left(i \nu_{y, s} \theta_{y}\right)\left\{1+\sum_{n=p} g_{n}\right\} \\
g_{n}=\frac{\left(\nu_{y, s}-\nu_{y}\right)}{\Delta \nu^{*}} \frac{2 \nu_{y} c_{n} \exp \left[-i(n-p) \theta_{y}\right]}{\left(n-\nu_{x}-\nu_{y}\right)(n-p)}
\end{gather*}
$$

$$
\begin{aligned}
\Delta \nu & =(1 / 4 \pi \rho) \int d s\left(\beta_{x} \beta_{y}\right)^{\frac{1}{2}} a_{1} \exp \left[i\left(-\nu_{x, s} \theta_{x}+\nu_{y, s} \theta_{y}\right)\right] \\
b_{n} & =\frac{1}{4 \pi \rho} \int d s\left(\beta_{x} \beta_{y}\right)^{\frac{1}{2}} a_{1} \exp \left[i\left(\left(n-\nu_{y}\right) \theta_{x}+\nu_{y} \theta_{y}\right)\right] \\
c_{n} & =\frac{1}{4 \pi \rho} \int d s\left(\beta_{x} \beta_{y}\right)^{\frac{1}{2}} a_{1} \exp \left[i\left(\nu_{x} \theta_{x}+\left(n-\nu_{x}\right) \theta_{y}\right)\right] \\
\theta_{x} & =\psi_{x} / \nu_{x}, \quad \theta_{y}=\psi_{y} / \nu_{y}
\end{aligned}
$$

$\nu_{x, s}$ and $\nu_{y, s}$ are the solutions of

$$
\begin{equation*}
\nu_{x, s}=\nu_{y, s}+p, \quad\left(\nu_{x, s}-\nu_{x}\right)=|\Delta \nu|^{2} \tag{25}
\end{equation*}
$$

There are two solutions of Eq. (25) corresponding to the two normal modes. For the mode for which $\nu_{x, s} \rightarrow \nu_{x}$ when $a_{1} \rightarrow 0$, we will put $\nu_{x, s}=\nu_{1}, \nu_{y, s}=\nu_{1}-p$. For the mode for which $\nu_{y, s} \rightarrow \nu_{y}$ when $a_{1} \rightarrow 0$, we will put $\nu_{y, s}=\nu_{2}, \nu_{x, s}=\nu_{2}+p$. The $A$ and $B$ coefficients are related by

$$
\begin{align*}
& B_{1}=\frac{-\left(\nu_{1}-\nu_{x}\right)}{\Delta \nu} A_{1} \text { for the } \nu_{1} \text { mode }  \tag{26}\\
& A_{2}=\frac{-\left(\nu_{2}-\nu_{y}\right)}{\Delta \nu^{*}} B_{2} \text { for the } \nu_{2} \text { mode }
\end{align*}
$$

The results for the eigenfunctions, Eq. (24) were found by solving the equations of motion to first order terms in $a_{1}$. It has been assumed that $\nu_{x}, \nu_{y}$ the unperturbed tune, is close to the coupling resonance $\nu_{x}=\nu_{y}+p$ and the $\nu_{x}-\nu_{y}-p$ can be considered to be small, of the same order as $a_{1}$. This last assumption allows the equations to be simplified and it is the case of most interest to us.

The $A$ and $B$ coefficients in Eq. (24) have now to be chosen so that the eigenfunctions are properly normalized, which means the eigenfunctions can be then expressed in terms of the orbit parameters like $\beta_{1}, \alpha_{1}, \psi_{1}$ and $\beta_{2}, \alpha_{2}, \psi_{2}$. To understand this better consider the 2 dimensional case. If we wish the eigenfunction to be related to $\beta, \psi$ by

$$
x=\beta^{1 / 2} \exp (i \psi)
$$

then

$$
p_{x}=x^{\prime}=\beta^{-1 / 2}(-\alpha+i) \exp (i \psi)
$$

and the two eigenfunctions are given by $x_{1}, x_{1}^{*}$ where

$$
x_{1}=\left[\begin{array}{c}
\beta^{\frac{1}{2}}  \tag{27}\\
\beta^{-\frac{1}{2}}(-\alpha+i)
\end{array}\right] \exp (i \psi)
$$

These eigenfunctions are normalized so that

$$
\begin{align*}
& \tilde{x}_{1}^{*} S x_{1}=2 i  \tag{28}\\
& S=\left[\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right]
\end{align*}
$$

$\tilde{x}_{1}$ is the transpose of $x_{1}$.
The normalization given by Eq. (28) gives the relationship between $x_{1}$ and $\beta, \alpha, \psi$ given by Eq. (27). It is shown in section $I$, that in the 4 dimensional case the normalization Eq. (28) will allow the eigenfunctions $x_{1}, x_{3}$ to be related to $\beta_{1}, \alpha_{1}, \psi_{1}$ and $\beta_{2}, \alpha_{2}, \psi_{2}$ in a corresponding way. In this case, $S$ is now the corresponding $4 \times 4$ matrix. Eq. (28) will be used to determine the coefficients $A, B$. This gives the relationship, see Ref. 7 where the following relationship is derived,

$$
\begin{equation*}
|A|^{2}\left(\nu_{x, s} / \nu_{x}\right)+|B|^{2}\left(\nu_{y, s} / \nu_{y}\right)=1 \tag{29}
\end{equation*}
$$

Eq. (29) together with Eq. (26) determine $A$ and $B$.
For the $\nu_{1}$ mode

$$
\begin{align*}
B_{1} & =-\frac{\nu_{1}-\nu_{x}}{\Delta \nu} A_{1} \\
\left|A_{1}\right|^{2} & \left(\frac{\nu_{1}}{\nu_{x}}+\frac{\nu_{1}-p}{\nu_{y}}\left|\frac{\nu_{1}-\nu_{x}}{\Delta \nu}\right|^{2}\right)=1 \tag{30}
\end{align*}
$$

For the $\nu_{2}$ mode

$$
\begin{align*}
A_{2} & =-\frac{\nu_{2}-\nu_{y}}{\Delta \nu^{*}} B_{2} \\
\left|B_{2}\right|^{2} & \left(\frac{\nu_{2}}{\nu_{y}}+\frac{\nu_{2}+p}{\nu_{x}}\left|\frac{\nu_{2}-\nu_{y}}{\Delta \nu}\right|^{2}\right)=1 \tag{31}
\end{align*}
$$

A case of particular interest is when the linear coupling has been corrected to make $\Delta \nu \simeq 0$. There are then two solutions of interest,

1. $|\Delta \nu| \ll\left|\nu_{x}-\nu_{y}-p\right|$
2. $\left|\nu_{x}-\nu_{y}-p\right| \ll|\Delta \nu|$

In case 1., $\Delta \nu$ has been made small enough so that the tune $\nu_{x}, \nu_{y}$ is well outside the width of the difference resonance. This may not always be achieved. $\nu_{x}, \nu_{y}$ may be very close to the difference resonance, and the best setting of the correction system to minimize the tune splitting does not have to correspond to $\Delta \nu=0$.

If $|\Delta \nu| \ll\left|\nu_{x}-\nu_{y}-p\right|$, one finds

$$
\begin{array}{ll}
\left|A_{1}\right|=1, & B_{1}=0  \tag{32}\\
\left|B_{2}\right|=1, & A_{2}=0 \\
\text { ppear to be decoupled. }
\end{array}
$$

The two modes appear to be decoupled.
If $\left|\nu_{x}-\nu_{y}-p\right| \ll|\Delta \nu|$, one finds

$$
\begin{align*}
& \left|A_{1}\right|=\left|B_{1}\right|=1 / \sqrt{2} \\
& \left|A_{2}\right|=\left|B_{2}\right|=1 / \sqrt{2} \tag{33}
\end{align*}
$$

The two modes appear completely coupled.
Examples of using the eigenfunction approach to compute the linear orbit parameters are given in Ref. 4,7.

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