One of Methods to Extract Truncated Taylor Map for Orbital and Spin Motion in Proton Storage Rings

V. Balandin
Institute for Nuclear Research of the Russian Academy of Sciences, Moscow

Abstract

The Taylor map coefficients obtained by any tracking code with differential algebra techniques accumulate two types of errors: computer rounding errors and errors defined by size of integration step. There is the approach free from defects of second type. That is the method of undetermined coefficients. Unfortunately it is not easy to produce computer code for the usually used form of this method. In this paper we present algorithm combining the advantages of method of undetermined coefficients and the calculating elegance of Lie algebraic techniques. The general approach using only the common properties of Lie groups and Hamilton's equations allows to obtain the truncated Taylor map for orbital and spin motion by the same way.

I. Introduction

The computer programs for analytical manipulations with polynomials have become an essential tool for the nonlinear design and analysis of circular accelerators [1], [2], [3], [4], [5]. The powerful methods as the numerical integration with using the differential algebra techniques and the direct summing the Lie exponent series with the machine precision allow to obtain the Taylor map coefficients up to the arbitrary hight order of the nonlinearity. The steps number in these methods depends on a desirable precision of the result and the length of computer word. In this paper the algorithm, the number of the steps in which does not depend on a precision, is described (like in the Gauss method of the matrix inversion the number of operations depends on the matrix dimension only). The main idea of this approach can be applied to the wide class of the hamiltonian and nonhamiltonian systems of ordinary differential equations. Lower this method is illustrated for the orbital motion through one magnet element (quadrupoles, sextupoles and etc.) in circular accelerators.

II. Realization for Orbital Motion

We will say that the function $g(z,t) \in HP(k)$ if $g(z,t)$ is the homogeneous polynomial of order k in respect to z with t-dependent polynomial coefficients.

Define the order of Taylor map truncation m and consider the initial Hamiltonian in the form

$$H^1(z,t) = H_2^1(z) + H_3^1(z,t) + \cdots + H_{m+1}^1(z,t) \qquad (1)$$

where $z = (x,y)$ are the canonical orbital variables ($dim(z) = 2n$) and $H_l^1 \in HP(l)$.

Introduce two additional canonical variables τ, L and new Hamiltonian

$$H^2(z,\tau,L) = H^1(z,\tau) + L$$

One obtain the autonomous differential equations system in the space of larger dimension

$$\frac{dx}{dt} = \frac{\partial H^2}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H^2}{\partial x}$$
$$\frac{d\tau}{dt} = \frac{\partial H^2}{\partial L}, \quad \frac{dL}{dt} = -\frac{\partial H^2}{\partial \tau}$$

All next steps consist of the sequence of canonical variables transformations:

a) So there is the symplectic $2n \times 2n$ matrix A that the second order part of the new Hamiltonian

$$H^3(z,\tau,L) = H^2(A \cdot z, \tau, L)$$

has the form:

$$H_2^3(z) = \bar{H}_2(z) + \tilde{H}_2(z)$$

$$\bar{H}_2(z) = \sum_{j=1}^{n} \lambda_j \cdot x_j \cdot y_j$$

$$\tilde{H}_2(z) = \frac{1}{2} \cdot \sum_{j=1}^{n} \sigma_j \cdot y_j^2 + \sum_{j=1}^{n-1} \epsilon_j \cdot x_j \cdot y_{j+1}$$

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where $\{\bar{H}_2(z), \tilde{H}_2(z)\} = 0$ ($\{\cdot,\cdot\}$ is the Poisson bracket) and

$$\epsilon = \begin{cases} 0 & \text{if } \lambda_j \neq \lambda_{j+1} \\ 1 & \text{otherwise} \end{cases}$$

$$\sigma = \begin{cases} 0 & \text{if } \lambda_j^2 + \epsilon_j^2 \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

(see ref. [6]).

b) Perform the Lie transformation of the Hamiltonian H^3 :

$$H^4(z,\tau,L) = exp(: -\tau\tilde{H}_2(z) :)H^3(z,\tau,L) =$$

$$= \tilde{H}_2(z) + L + \sum_{l=3}^{m+1} H_l^3(exp(: -\tau\tilde{H}_2(z) :)z,\tau)$$

It is possible to proof directly that the series

$$exp(: -\tau\tilde{H}_2(z) :)z$$

contains the finite number of members only. We note also the way which can be applied to non-hamiltonian equations.

Consider the system

$$\frac{dw}{dt} = (D+N)\cdot w + f(w,t)$$

where N is a nilpotent matrix ($N^{k+1} = 0$), $D \cdot N = N \cdot D$, $f(w,t) = f_2(w,t) + \cdots + f_m(w,t)$, $f_l(w,t) \in HP(l)$.

Using the coordinate transformation

$$w = exp(t\cdot N)\cdot u = (I + t\cdot N + \cdots + \frac{t^k}{k!}\cdot N^k)\cdot u$$

one obtain

$$\frac{du}{dt} = D\cdot u + g(u,t) =$$

$$= D\cdot u + exp(-t\cdot N)\cdot f(exp(t\cdot N)\cdot u, t)$$

Here a function $g(u,t)$ has the same structure as a function $f(w,t)$. i. e. $g(u,t) = g_2(u,t) + \cdots + g_m(u,t)$ and $g_l(u,t) \in HP(l)$.

So we obtain after steps *a*) and *b*) the Hamiltonian with the diagonal second order part.

c) This part consists of $m-1$ steps of nonlinear coordinate transformations $l = 3, 4, \cdots, m+1$

$$H^{l+2}(z,\tau,L) = exp(: W_l(z,\tau) :)H^{l+1}(z,\tau,L)$$

where $W_l(z,\tau) \in HP(l)$.

If $W_l(z,\tau)$ satisfies the equation

$$\frac{\partial W_l}{\partial \tau} + \{W_l, \tilde{H}_2\} + H_l^{l+1} = N_l \qquad (2)$$

then the final Hamiltonian has the form

$$N(z,\tau,L) = H^{m+3}(z,\tau,L) =_{m+1}$$

$$=_{m+1} \bar{H}_2 + L + N_3(z,\tau) + \cdots + N_{m+1}(z,\tau) \qquad (3)$$

where $=_{m+1}$ indicates that the right and left side agrees up to order $m+1$.

The equation (2) is divided in independent equations for coefficients of monomials $x^I \cdot y^J$. Each of them has the form:

$$\frac{dw}{d\tau} + \mu\cdot w = n - h \qquad (4)$$

where $\mu = (I-J)\cdot\lambda$. For any given polynomials

$$h(\tau) = h_0 + h_1\cdot\tau + \cdots + h_k\cdot\tau^k$$
$$n(\tau) = n_0 + n_1\cdot\tau + \cdots + n_k\cdot\tau^k$$

the equation (4) has the polynomial solution. If $\mu = 0$ then

$$w(\tau) = const +$$

$$+(n_0 - h_0)\cdot\tau + \cdots + \frac{1}{k+1}\cdot(n_k - h_k)\cdot\tau^{k+1}$$

else

$$w(\tau) = w_0 + w_1\cdot\tau + \cdots + w_k\cdot\tau^k$$

$$\begin{cases} w_k = \frac{1}{\mu}\cdot(n_k - h_k) \\ w_{k-1} = \frac{1}{\mu}\cdot(n_{k-1} - h_{k-1} - k\cdot w_k) \\ \cdots \\ w_0 = \frac{1}{\mu}\cdot(n_0 - h_0 - w_1) \end{cases}$$

d) Using the notation

$$: C(z,\tau) :=$$

$$= exp(: W_{m+1} :)\cdots exp(: W_3 :)exp(: -\tau\tilde{H}_2 :) : A :$$

we obtain finally

$$: C(z,\tau) : H^2(z,\tau,L) =_{m+1} N(z,\tau,L)$$

$$exp(: -H^2 :) =_m : C :^{-1} exp(: -N :) : C : \qquad (5)$$

e) In mathematical sense the equation (2) has the solution for any $N_l \in HP(l)$. Consider the important particular case $N_l = N_l(z)$ and $N = \bar{N}(z) + L$. In this case for any real α we obtain:

$$exp(: -H^2 :) =_m exp(: \alpha L :) : C(z,\tau+\alpha) :^{-1} \cdot$$

$$\cdot exp(: -\bar{N}(z) :) : C(z,\tau+1+\alpha) : exp(: -(1+\alpha)L :) \qquad (6)$$

If the initial Hamiltonian $H^1 = H^1(z)$ then from (6) the important representation follows

$$exp(: -H^1 :) =_m$$

$$=_m : C(z, \tau + \alpha) :^{-1} exp(: -\bar{N}(z) :) : C(z, \tau + 1 + \alpha) : \quad (7)$$

and in z-space we have

$$exp(: -H^1(z) :) =_m$$

$$=_m : C(z, \alpha) :^{-1} exp(: -\bar{N}(z) :) : C(z, 1 + \alpha) : \quad (8)$$

If $\{\bar{N}(z), \bar{H}_2(z)\} = 0$ then for any given order of Taylor map truncation m the right side in (8) can be calculated for the finite number of operations with polynomials.

The representation (8) includes the normal form decomposition in the nonresonance case [7]. It also includes the more unusual factorization with $\bar{N}(z) = \bar{H}_2(z)$ which exists for any initial autonomous Hamiltonian $H^1(z)$ independently of properties λ_j. If we use the possibility of the arbitrary choice of $N_l(z)(l = 3, \cdots, m + 1)$ we can obtain the another interesting results from formula (8).

III. Computer Implementation

What are more difficult steps in the computer realization of the described above algorithm? For the arbitrary initial autonomous Hamiltonian it is the finding the matrix A in the point a) and the process of the obtaining the solution of the equation (4) in the case when μ does not equal zero but very small. Fortunately, we never meet this situation when we consider magnet elements like ideal quadrupoles, sextupoles and other multipoles. The method of the map calculation with help of the representation (8) has been implemented in the computer code VasiLIE [3]. To the author surprise the calculation speed of this algorithm is not so slow as it seems before the realization. It is near the same as the speed of the direct summing Lie exponent series with the machine precision (the tests were performed using IBM PC 386 computer for maps of 4 variables and orders $6 - 10$ and $\alpha = 0, -0.5$).

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