Multi-dimensional Beam Emittance and β-functions

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Abstract

The concept of r.m.s. emittance is extended to the case of several degrees of freedom that are coupled. That multi-dimensional emittance is lower than the product of the emittances attached to each degree of freedom, but is conserved in a linear motion. An envelope-hyperellipsoid is introduced to define the β-functions of the beam envelope. On the contrary of an one-degree of freedom motion, it is emphasized that these envelope functions differ from the amplitude functions of the normal modes of motion as a result of the difference between the Liouville and Lagrange invariants.

1. INTRODUCTION

A statistical definition of beam emittance has been originally introduced by P. Lapostolle [1] in a 2-dimensional phase space of an one-degree of freedom motion. The statistical point of view is the most natural way to study the particle spread in phase space. It has been reviewed in [2] and it is here extended to the case of several degrees of freedom that are coupled. The key role is here played by the covariance matrix of the particle coordinates. The emittance is proportional to the square root of its determinant. It is also involved in the expressions of the Liouville and Lagrange invariants that characterize a linear motion.

2. THE STATISTICAL DEFINITION OF MULTIDIMENSIONAL EMITTANCE

The second-order moments give the main statistical characteristics of a set of points in a 2p-dimensional phase space (p is the number of degrees of freedom). With respect to a frame, the origin of which is taken at the barycentre of the points, the moment $<x_\alpha x_\beta>$ is obtained by averaging the coordinate product $x_\alpha x_\beta$ over the set of points (x_α, x_β are two coordinates of one point : $\alpha, \beta = 1,...,2p$). The second-order moments are embodied in the covariance matrix V :

$$V = \begin{pmatrix} <x_1 x_1> & \dots & <x_1 x_{2p}> \\ \dots & \dots & \dots \\ <x_{2p} x_1> & \dots & <x_{2p} x_{2p}> \end{pmatrix}$$

Hereafter, it is convenient to write the covariance matrix as the statistical average of a formal product :

$$V = \langle \mathbf{x}.\tilde{\mathbf{x}} \rangle$$

where $\mathbf{x}$ is a coordinate column-vector and $\tilde{\mathbf{x}}$ is the transposed coordinate row-vector.

The covariance matrix V is real and symmetric. It can always be diagonalized by a similarity transform, defined by an orthogonal matrix Ω ($\Omega\tilde{\Omega} = 1$). Assuming that the coordinate frame is orthogonal, that transform corresponds to a change of orthogonal frame with respect to which the new coordinate vector is $\mathbf{X} = \Omega\mathbf{x}$. One recognizes that the new matrix $W = \Omega V \Omega^{-1}$ is the covariance matrix w.r.t. the new frame :

$$W = \Omega \langle \mathbf{x}.\tilde{\mathbf{x}} \rangle \Omega^{-1} = \Omega \langle \mathbf{x}.\tilde{\mathbf{x}} \rangle \tilde{\Omega} = \langle \Omega\mathbf{x}.\widetilde{\Omega\mathbf{x}} \rangle = \langle \mathbf{X}.\tilde{\mathbf{X}} \rangle$$

The covariance matrix W w.r.t. the new coordinate frame is :

$$W = \begin{pmatrix} \langle X_1^2 \rangle & 0 & \cdots & 0 \\ 0 & \langle X_2^2 \rangle & \cdots & 0 \\ 0 & 0 & \cdots & \langle X_{2p}^2 \rangle \end{pmatrix}$$

The diagonal element $<X_\alpha^2>$ is the mean square distance to the hyperplane perpendicular to the frame axis OX_α. The square root $\sqrt{< X_\alpha^2 >}$ is the corresponding r.m.s. distance $\sigma(X_\alpha)$.

To measure the spread of points in phase space, it is natural to define the multidimensional emittance ε_p as :

$$\varepsilon_p = 2\sigma(X_1).2\sigma(X_2)\cdots 2\sigma(X_{2p})$$

i.e. :

$$\varepsilon_p = 2^{2p} \sqrt{\det(V)}$$

where one has used the property that V and W have the same determinant, in order to express the emittance ε_p as function of the second-order moments w.r.t. the original frame.

In a two-dimensional phase space that definition reproduces the so-called r.m.s. emittance, first introduced by P. Lapostolle [1] :

$$\varepsilon_1 = 4 \sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$

The numerical factor 2^{2p} is optional. It is just introduced to give a realistic and quantitative measure of the volume occupied by the points in the phase space.

It is worth noting that the mean distances $\langle X_\alpha^2\rangle$ are the eigenvalues of the covariance matrix V and that these eigenvalues are never negative (V is a semi-definite positive matrix). Conversely, any matrix real, symmetric, and semi-definite positive, can be considered as a covariance matrix. Effectively, its positive eigenvalues can be taken as the mean square distances to the hyperplanes perpendicular to the axes of a particular coordinate frame.

Normally, a 2p-dimensional phase space is the product of p two-dimensional subspaces, each one being the phase space for one degree of freedom of the particle motion. For instance, the product of the two-dimensional phase space for the x-transverse motion, of the two-dimensional phase space for the y-transverse motion and of the two-dimensional phase space for the z-longitudinal motion is a six-dimensional phase space. Using a generalization [3] of the Hadamard determinant inequality, it can be shown that the 2p-dimensional emittance ε_p cannot be larger than the product of the emittances $\varepsilon^{(1)}$, $\varepsilon^{(2)}$,...,$\varepsilon^{(p)}$ in the p two-dimensional subspaces :

$$\varepsilon_p \le \varepsilon^{(1)}\,\varepsilon^{(2)} \cdots \varepsilon^{(p)}$$

The equality only occurs when the degrees of freedom are uncorrelated, i.e. when all the correlation moments $\langle x_\alpha x_\beta\rangle$ of two coordinates x_α, x_β corresponding to two different degrees of freedom are vanishing. Usually, a beam is said coupled when the degrees of freedom are correlated. Accordingly, the correlation moments between them will be hereafter named coupling moments. For instance, in the case of the 4-dimensional transverse phase space with coordinates x, x', y, y', there are four such coupling moments : $\langle xy\rangle$, $\langle xy'\rangle$, $\langle x'y\rangle$, $\langle x'y'\rangle$. The 4-dimensional emittance ε_2 is the product of the emittances ε_x, ε_y, of the x and y transverse motions, only if these four coupling moments vanish.

The preceding inequality geometrically means that the volume occupied by the points in the 2p-dimensional phase space is less than the product of the areas occupied on each two-dimensional subspace, apart when they are uncoupled.

It may even happen that the emittance ε_p vanishes although none of the emittances $\varepsilon^{(1)}$, $\varepsilon^{(2)}$, ... , $\varepsilon^{(p)}$ vanishes. That occurs when two coordinates x_α, x_β corresponding to two different degrees of freedom are linearly dependent, i. e. fully correlated. For instance, in the case of the 4-dimensional transverse phase space with coordinates x, x', y, y', it occurs if the coordinates x and y would be proportional.

3. EMITTANCE CONSERVATION AND INVARIANTS IN LINEAR MOTION

In a linear motion, governed by a quadratic hamiltonian, the conservation of the multidimensional emittance results from the Liouville theorem. If R is the linear and symplectic mapping that connects the particle coordinates at time t to the coordinates at initial time t_0, the covariance matrix V is mapped according to :

$$V(t) = \langle x(t).\tilde{x}(t)\rangle = \langle Rx(t_0).\widetilde{Rx}(t_0)\rangle = RV(t_0)\,\tilde{R}$$

The Liouville theorem implies det(R) = 1 and det(V) is constant.

Moreover, the geometrical shape of the particle spread in phase space can be made more precise by defining an envelope-hyperellipsoid *H*. With respect to the coordinate frame corresponding to the diagonal covariance matrix W, the equation of *H* is :

$$\sum_{\alpha=1}^{2p} \frac{X_\alpha^2}{4\langle X_\alpha^2\rangle} = 1$$

either, with a matrix notation :

$$\tilde{X} W^{-1} X = 4$$

Returning to the normal coordinate frame by an orthogonal transformation, the equation of *H* keeps the same form :

$$\tilde{x} V^{-1} x = 4$$

The volume $\Omega(H)$ of the envelope-hyperellipsoid H is proportional to the emittance ε_p :

$$\Omega(H) = \frac{\pi^p}{\Gamma(p+1)} \varepsilon_p$$

and the conservation of the emittance expresses the conservation of the volume.

The projection of the envelope-hyperellipsoid H on any 2-dimensional subspace, as x, x', is the envelope-ellipse describing the geometrical shape of the particle spread in that subspace. Its equation is [2] :

$$x^2 \langle x'^2 \rangle - 2xx' \langle xx' \rangle + x'^2 \langle x^2 \rangle = \frac{\varepsilon_x^2}{4}$$

In a coupled motion the 2-dimensional emittance ε_x is not constant. To define, as usual, the envelope-functions β_x, α_x and γ_x of the beam in the x,x' subspace, one must use the invariant emittance ε_p instead of the emittance ε_x in that subspace :

$$\beta_x \varepsilon_p = 4\langle x^2 \rangle$$

$$\gamma_x \varepsilon_p = 4\langle x'^2 \rangle$$

$$\alpha_x \varepsilon_p = -4\langle xx' \rangle$$

with the relations :

$$\alpha_x = -\frac{1}{2} \frac{d\beta_x}{ds}$$

$$\beta_x \gamma_x - \alpha_x^2 = \left(\frac{\varepsilon_x}{\varepsilon_p}\right)^2$$

These envelope-functions, defined in each subspace, together with the coupling moments completely characterize the beam evolution in phase space.

Now, the LHS of the envelope-hyperellipsoid equation : $\tilde{x}\, V^{-1}\, x$ is a quadratic form left invariant by any linear mapping. It is the Liouville invariant expressing the hypervolume conservation in phase space. Another invariant quadratic form can be obtained from the Lagrange invariant [4] :

$$\tilde{x}_i\, \eta\, x_j$$

where x_i and x_j are the coordinate vectors of two particles i and j, and η is the symplectic unit matrix. Squaring that invariant and averaging over the particle j, one obtains the invariant quadratic form :

$$\tilde{x}\, \eta\, V\, \eta\, x$$

For instance, in the 4-dimensional phase space of the coupled transverse betatron motion, these two invariant quadratic forms determine two hyperellipsoids. The particle moves on their intersection that is a bidimensional torus. As well-known, that motion of an individual particle is characterized by two frequencies ω_1, ω_2 and two amplitude functions β_1, β_2. These two amplitude functions are different from the beam envelope-functions β_x, β_y defined above. They become identical only in the case of an uncoupled motion. It is due to the fact that the Liouville invariant and the Lagrange invariant are identical in the case of an one-degree of freedom motion.

4. REFERENCES

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