Even Order Snake Resonances

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ABSTRACT

We found that the perturbed spin tune due to the imperfection resonance plays an important role in beam depolarization at snake resonances. We also found that *even* order snake resonances exist in the overlapping intrinsic and imperfection resonances. Due to the perturbed spin tune shift of imperfection resonances, each snake resonance splits into two.

1 Introduction

The spin equation of motion for a spin particle, governed by the magnetic interaction between the magnetic dipole moment of the particle and the static magnetic field in a synchrotron, is given by the Thomas-BMT equation [1], $\frac{d\vec{S}}{dt} = \frac{e}{\gamma m}\vec{S} \times [(1+G\gamma)\vec{B}_\perp + (1+G)\vec{B}_{\|}]$, where $\vec{B}_\perp$ and $\vec{B}_{\|}$ are the transverse and longitudinal components of the magnetic fields with respect to the velocity vector, $\vec{\beta}$. In a planar synchrotron, vertical magnetic fields are needed to guide the orbiting particle around a closed path. Thus the spin vector is precessing with respect to the vertical axis at a frequency $G\gamma f_0$, where f_0 is the revolution frequency, $G = \frac{g}{2} - 1$ is the anomalous magnetic g-factor and γ is the relativistic Lorentz factor. The quantity, $G\gamma$, representing the number of spin precessions per revolution, is called the spin tune.

In a synchrotron, strong quadrupole fields are also needed to focus the beam to a small size. Those particles moving off-center vertically in quadrupoles experience horizontal fields, which will kick the spin vector away from the vertical axis. Since quadrupole magnets and the particle closed orbits are periodic in a circular accelerator and the betatron and the synchrotron motions are quasiperiodic, perturbing kicks to the spin vector can be decomposed into harmonics, K, given by $K = n + m\nu_z + \ell\nu_x + k\nu_{syn}$, where ν_z, ν_x and ν_{syn} are respectively the vertical betatron, the horizontal betatron and the synchrotron tunes, and k, ℓ, m, n are integers. The imperfection resonances, due to the vertical closed orbit errors, are located at integer harmonics, $K = n$. The intrinsic resonances, due to the vertical betatron motion, are located at $K = nP + \nu_z$, where P is the superperiodicity of the accelerator. Other depolarizing resonances arise from linear or nonlinear betatron coupling, vertical dispersion, synchro-betatron coupling and random field errors. When the spin precession frequency is in phase with the harmonics of perturbing kicks, i.e. $G\gamma = K$, these spin perturbing kicks add up coherently every turn around the ring. Therefore the beam can be depolarized.

To avoid a spin resonance condition, Derbenev and Kondratenko [3] proposed to use a local spin rotator, which rotates the spin vector 180° about an axis in the horizontal plane. These spin rotators are called snakes. Using snakes in an accelerator, the spin tune of the particle can become $\frac{1}{2}$ and independent of energy. The resonance condition can be avoided.

2 Snakes and Spin Motion

Snakes are local spin rotators, which rotate particle spin by π radians about a horizontal axis locally without perturbing particle orbits outside a snake region. A partial snake differs only in the amount of spin rotation angle, e.g. a 10% snake rotates spin by 0.1π radians. Thus a snake is characterized by the amount of *spin rotation angle*, ϕ, and the *snake axis angle*, ϕ_s, with respect to $\hat{e}_1$ (radially outward direction). The spinor wave function at a snake will be transformed locally according to $\Psi(\theta^+) = e^{-i\frac{\phi}{2}\hat{n}_s\cdot\vec{\sigma}}\Psi(\theta^-)$, where ϕ is spin rotation angle and $\hat{n}_s = (\cos\phi_s, \sin\phi_s, 0)$ denotes the snake axis with respect to radially outward direction, $\hat{e}_1$. $\theta^\pm$ depict azimuthal orbit rotation angles just before and after the snake. More specifically, at $\phi = \pi$, or the 100% snake, the spinor wave function can be transformed as,

$$\Psi(\theta^+) = e^{-i\frac{\pi}{2}\hat{n}_s\cdot\vec{\sigma}}\Psi(\theta^-) = T_s(\phi_s)\Psi(\theta^-), \tag{1}$$

where $T_s(\phi_s) = -i\hat{n}_s \cdot \vec{\sigma}$ is the spin transfer matrix for a 100% snake.

Let us consider a perfect circular accelerator with two snakes, $-i\sigma_1, -i\sigma_2$, separated by π orbital angle apart. The one turn spin transfer matrix (OTM) is given by

$$[-i\sigma_2]e^{-i\frac{G\gamma\pi}{2}\sigma_3}[-i\sigma_1]e^{-i\frac{G\gamma\pi}{2}\sigma_3} = i\sigma_3. \tag{2}$$

Thus the spin tune, obtained from the trace of the one turn spin transfer matrix, is $\frac{1}{2}$ and the stable spin closed orbit is vertical. Now we introduce a small constant local spin angular precessing kick, χ, about an axis $\hat{n}_k$ in the horizontal plane, the spin transfer matrix becomes,

$$T_1 = e^{-i\frac{\chi}{2}\hat{n}_k\cdot\vec{\sigma}}i\sigma_3. \tag{3}$$

Because $\hat{n}_k$ is in the horizontal plane, the evolution of the spin transfer matrix at the nth revolution becomes,

$$T^{(n)} = [T_1]^n = \begin{cases} [i\sigma_3]^n & \text{if } n = \text{even} \\ T_1[i\sigma_3]^{(n-1)} & \text{if } n = \text{odd} \end{cases}, \tag{4}$$

which means that the perturbed spin precessing kicks cancel each other every two turns around the accelerator. Thus the snake is effective in correcting imperfection resonances due to a localized constant spin perturbing kick.

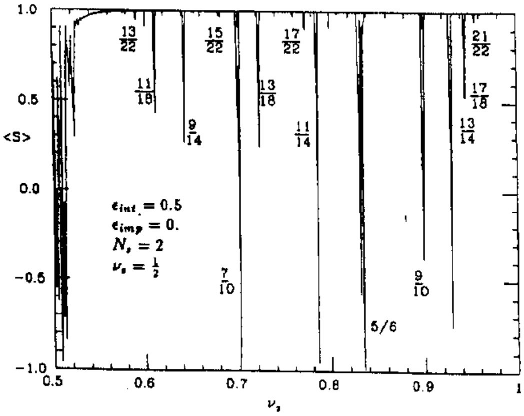

Figure 1: The vertical polarization after passing through an intrinsic depolarization resonance with two snakes is plotted as a function of the vertical betatron tune ν_z.

Extending the model a step further, we assume that the precessing kick is different in each turn, the spin transfer matrix becomes,

$$T^{(n)} = \prod_{m=1}^{n} T_m = e^{-i\frac{1}{2}[\sum_{m=1}^{n}(-1)^{n-m}\chi_m]\hat{n}_k\cdot\vec{\sigma}}[i\sigma_3]^n. \quad (5)$$

The vertical spin vector is given by,

$$S_3^{(n)} = 1 - 2\sin^2[\tfrac{1}{2}\textstyle\sum_{m=1}^{n}(-1)^{n-m}\chi_m].$$

Now if the spin perturbation kicks are due to a betatron motion, these kicks are correlated by $\chi_m = \chi_0 \cos 2m\pi\nu_z$, where ν_z is the fractional part of the vertical betatron tune. When the vertical betatron tune is $\nu_z = \frac{1}{2}$, each kick adds up coherently. The spin vector will precess around the $\hat{n}_k$ axis at a precessing tune of $\frac{\chi_0}{2\pi}$, i.e. it takes $\frac{2\pi}{\chi_0}$ turns to complete one revolution around the $\hat{n}_k$ axis.

3 Odd order snake resonances

Subsequent studies show that when the resonance strength is large, new spin depolarizing resonances occur at some fractional betatron tunes. These resonances are called *snake resonances* [4] . Snake resonances, due to coherent higher order spin perturbing kicks, are located at

$$\nu_s + \ell K = \text{ integer}, \quad \ell = 1, 3, 5, 7, \cdots, \quad (6)$$

where ν_s is the spin tune and K is the spin depolarizing resonant harmonic. For $\nu_s = \frac{1}{2}$, we expect that snake resonances occur at the following fractional betatron tunes, $\nu_z = \frac{1}{2}, \frac{1}{6}, \frac{5}{6}, \frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}, \frac{1}{14}, \frac{3}{14}, \cdots$, where the lowest order snake resonance has been observed [5] . Other higher order snake resonances have been identified in numerical simulation (Fig. 1). It is interesting to note that the numerical simulations show no apparent even order snake resonances at

$$\nu_s + \ell K = \text{ integer}, \quad \ell = 2, 4, 6, 8, \cdots. \quad (7)$$

Several reasons for the nonexistence of even order snake resonances were given in the past [4,6], which has never been tested in the case of overlapping resonances.

Overlapping resonances are important in high energy accelerators. [6,7] An important imperfection resonance occur usually at the integer nearest to the dominant intrinsic resonance. Therfore overlapping intrinsic and imperfection resonances constitute the most important problem in the spin dynamics during polarized proton acceleration.

4 Even Order Snake Resonances

To understand the effect of imperfection resonances on the spin motion, we reduce intrinsic resonance strength in our calculation to $\epsilon_{int} = 0.137$, where only low order snake resonances at $\nu_z = 1/2,\ 1/6,\ 5/6$ are important. When an imperfection resonance at $\epsilon_{imp} = 0.13$ is included, we found that even order snake resonances at $\nu_z = 3/4,\ 5/8,\ 7/8, \cdots$ appear. Furthermore, all snake resonances split into double peaks shown in Fig. 2. The distance of these two peaks increases with the strength of the imperfection resonance. Note that the even order snake resonance becomes more important than the odd order snake resonance and the odd order snake resonance is not affected by the imperfection resonance. Note also that double peaks occur for each snake resonance. The feature of double peaks can be understood easily knowing that the imperfection resonance generates a perturbed spin tune shift. The snake resonance condition becomes

$$\frac{1}{2} + \Delta Q_s \pm \ell\nu_z = \text{ integer}, \quad \ell = \text{ integer}, \quad (8)$$

where ΔQ_s is the perturbed spin tune shift from the imperfection resonance given by

$$|\Delta Q_s| \approx \frac{1}{\pi}\arcsin[\sin^2\frac{\pi\epsilon_{imp}}{N_s}], \quad (9)$$

where the actual magnitude and sign of the spin tune shift depend on the closed orbit of the circular accelerator. The distance of splitting becomes smaller at higher order snake resonances (Fig. (2)) with $\Delta\nu_z = \pm\frac{1}{\ell}\Delta Q_s$. The depolarization line shape of these double peak reflects the important effect of perturbed spin tune shift on the snake resonances at the maximum spin tune shift.

To understand the essential mechanism of the even order snake resonances in the presence of overlapping spin resonances, we consider the model of the spin transfer matrix. The OTM of the overlapping intrinsic and imperfection resonances can be expressed as

$$\tilde{\tau} = e^{-i\frac{\chi}{2}\sigma_1}\tau(\theta_0 + 2\pi, \theta_0), \quad (10)$$

where $\tau(\theta_0 + 2\pi, \theta_0)$ is the OTM without imperfection resonance and we have assumed a small local spin precessing kick, χ, about the $\hat{e}_1$ axis. The resonance strength of the imperfection resonance is given by $\epsilon_{imp} = \chi/2\pi$ at all integer harmonics. Due to the imperfection resonance, the off-diagonal matrix elements now contain a term oscillating at

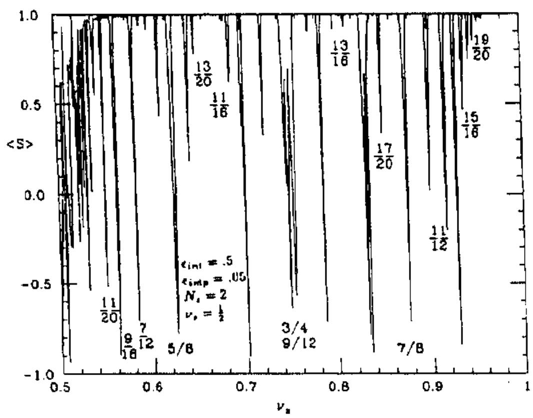

Figure 2: Vertical polarization after passing through an overlapping intrinsic and imperfection resonances with two snakes

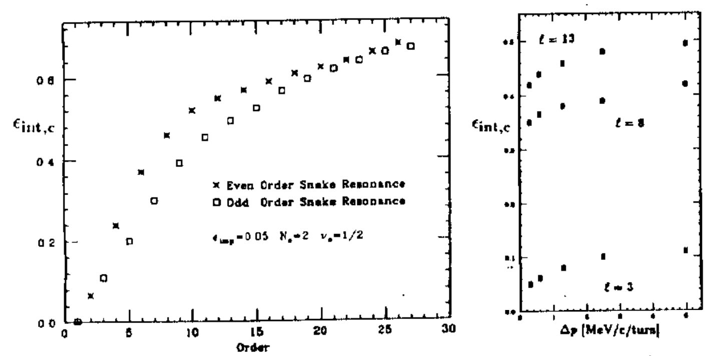

Figure 3: The critical snake resonance strength vs the order of snake resonance is plotted for $\dot{p}_0 = 5$ MeV/c per turn on the left. The critical resonance strength vs the acceleration rate is plotted in the middle part and the snake resonance strength for the even order resonance as a function of the imperfection resonance is shown on the right

two times the betatron frequency with an amplitude proportional to $b^2 \sin \frac{\chi}{2}$. Thus the snake resonance condition is given by $\nu_s \pm 2K =$ integer. Performing similar higher order analysis, one obtain all even order snake resonances.

5 Critical Resonance Strength

Let us define the critical resonance strength as the resonance strength that the polarization is preserved to within 1.5% of full polarization. Fig. 3 shows the critical resonance strength for the odd and even order snake resonances at the acceleration rate of 5 MeV/c per revolution. Depending on the acceleration rate, the critical resonacne strength can be described by the following formula fitted to numerical simulation results,

$$\begin{aligned}
\epsilon_{c,5/6} &= \ln[1.12(\frac{\dot{p}}{\dot{p}_0})^{0.024}], \\
\epsilon_{c,21/26} &= \ln[1.64(\frac{\dot{p}}{\dot{p}_0})^{0.024}], \\
\epsilon_{c,13/16} &= \ln[1.50(\frac{\dot{p}}{\dot{p}_0})^{0.020}],
\end{aligned}$$

where the reference acceleration rate is $\dot{p}_0 = 5$ MeV/c per revolution. Here we study only $K = \frac{5}{6}, \frac{13}{16}, \frac{21}{26}$ snake resonances with the assumption that the betatron tune are chosen to lie in between $\frac{4}{5}$ and $\frac{5}{6}$ for example for RHIC at BNL.

6 Conclusions

We found that snake resonances, located at $\nu_s + \ell K =$ integer, are the major source of depolarization in synchrotrons with snakes, where the integer ℓ is called the order of snake resonance, K is the spin resonance harmonics. When imperfection resonances are overlapping with intrinsic resonances, even order snake resonances appear. The perturbed spin tune, arising from imperfection resonances, plays an essential role in the depolarization mechanism, it causes each snake resonance to split into two resonances.

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