

DIFFUSION PHENOMENA IN SIMPLE HAMILTONIAN SYSTEMS: SOME ANALYTICAL AND NUMERICAL RESULTS

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Abstract

We study both numerically and analytically some simple Hamiltonian systems perturbed by a random noise or by a periodic (or quasi-periodic) noise. In this way we simulate the effects of the ripple in the power supply on the betatronic motion in a particle accelerator. We consider the dependence of the diffusion in the phase space on the relevant parameters of our system like the nonlinear terms, the strength of the noise and, in the deterministic case, its modulation frequency. We discuss also the possibility of describing the evolution of a distribution function for an integral of motion of the unperturbed system, like the action or the energy, by means of a Fokker-Planck equation. The results are compared with numerical simulations.

I. INTRODUCTION

Recent experiments in high energies hadron colliders have shown that the beam lifetime is substantially decreased when the nonlinear effects due to the multipolar errors are combined with the fluctuations of current (ripples) [1]. At present no satisfactory interpretation of the experimental results has been found and no simple models have been extensively investigated. In the experiments the slow periodic modulation is enhanced but it is not evident that the effects of a stochastic modulation can be a priori neglected; on the other hand when the dynamics is almost linear, no appreciable diffusion is observed and the beam is stable. This suggests that, if a stochastic modulation is present it affects the phase rather than the amplitude of the betatronic oscillations; the diffusion would then depend uniquely on the coupling with the nonlinear terms. We analyze here a simple model of a nonlinear integrable hamiltonian system where, the frequency is modulated with a stochastic perturbation or periodic perturbation. The phase space of the unperturbed hamiltonian has a separatrix which corresponds to the dynamic aperture. The presence of modulation allows the orbits to reach the separatrix and escape to infinity in a finite time interval. We look for the time evolution of the distribution function for a given initial population. Even in this oversimplified model the numerical simulations are extremely heavy and

it is hard to investigate the dependence on the parameters. The behaviour of the stochastic and periodic modulation is radically different. In the first case we justify theoretically the description of the diffusion in the nonlinear invariant of motion, by a Fokker-Planck equation [2] with a variable diffusion coefficient. The presence of the separatrix (dynamic aperture) is taken into account with an absorbing barrier, whereas a reflecting boundary condition (flux conservation) is imposed at the origin. The presence of a sextupole leads to a cubic diffusion coefficient which makes the diffusion extremely slow close to the origin, whereas it becomes significant in the vicinity of the separatrix where the absorption by the barrier simulates the escape to infinity.

The slow periodic modulation is analyzed in the framework of the adiabatic theory [3]. The origin is surrounded by invariant domains which are swept by any orbit when the frequency is slowly varied. On the contrary the region swept by the separatrix is chaotic and once the outer boundary is reached the escape to infinity still occurs. Compared to the stochastic modulation, there is a region defined by the inner boundary of the pulsating separatrix which is stable and the evolution of the distribution function in the chaotic region is not of diffusive type.

II. MODEL AND RESULTS

We consider the Hamiltonian

$$H = \omega \frac{p^2 + x^2}{2} (1 + \epsilon \xi(t)) - \frac{x^3}{3} \quad (1)$$

which models the betatronic motion in the horizontal plane in the presence of sextupoles using normalized coordinates. In (1) $\xi(t)$ denotes continuous realization of a random stationary process with zero average $\langle \xi \rangle = 0$ and correlation $G(t - t') \equiv \langle \xi(t) \xi(t') \rangle$ and ϵ is a small parameter. We will then consider the limit case in which the correlation length vanishes and the process becomes δ correlated; such a limit describes the increments of a Wiener process.

Introducing the action angle variables for the harmonic oscillator $x = \sqrt{2j} \sin \theta$ and $p = \sqrt{2j} \cos \theta$ we have

$$H = \omega j - \frac{2^{3/2}}{3} j^{3/2} \sin^3 \theta + \epsilon \omega j \xi(t) \equiv h_0(j, \theta) + \epsilon \omega j \xi(t) \quad (2)$$

By solving exactly the Hamilton's Jacobi equation for the hamiltonian h_0 or approximately by perturbation theory we determine a unique transformation $j = V(\theta, J)$ close to identity such that $h_0(j, \theta) = H_0(J)$ and

$$H = H_0(J) + \epsilon V(J, \theta) \xi(t) \quad (3)$$

We remark that we have chosen to work with the initial angle since the simulations are carried out choosing a uniform distribution in the initial angle θ .

Following a procedure introduced by Gurievich et al. [4] in plasma physics, we separate the phase space density, which satisfies Liouville's equation, into an average and a fluctuating part

$$\rho = \rho_0 + \epsilon \rho_1, \quad \rho_0 = \langle \rho \rangle \quad \langle \rho_1 \rangle = 0 \quad (4)$$

Choosing an initially uniform distribution in the angle or for t large enough that a uniform distribution is attained, ρ_0 will remain independent of Θ and will satisfy the Fokker-Planck equation

$$\frac{\partial \rho_0}{\partial t} = \frac{\epsilon^2}{2} \frac{\partial}{\partial J} \langle V_\theta^2 \rangle \frac{\partial \rho_0}{\partial J} + O(\epsilon^3) \quad (5)$$

which is the diffusion equation for the action. The diffusion coefficient

$$D_J = \frac{\epsilon^2}{2} \langle V_\theta^2 \rangle \quad (6)$$

depends on the action itself and agrees with the quasilinear approximation.

The derivation is based on the Liouville equation and an expansion for ϵ small; the hypothesis of a δ correlated noise is crucial to obtain a differential equation like (5) rather than an integro-differential equation. Even limit theorems in the theory of stochastic processes do not allow to recover a Fokker-Planck equation if the δ correlation hypothesis is dropped.

Even though the exact computation of $H_0(J)$ and $V(J, \theta)$ could be carried out we preferred to compute the second order in perturbation theory since this is the only available way of determining $D(J)$ in more realistic models. According to the canonical perturbation theory we have:

$$H(J) = \omega J - \frac{1}{\omega} \frac{5}{12} J^2 \quad (7)$$

and the diffusion coefficient reads

$$D = \frac{\epsilon^2}{2} \langle V_\theta^2 \rangle = \frac{\epsilon^2}{2} \left[J^3 + \frac{21}{8} \frac{J^4}{\omega^2} \right] \quad (8)$$

The position of the absorbing barrier $J = J_s$ is computed by imposing $H_0(J_s) = E_s$ where $E_s = \omega^3/6$ is the energy of the separatrix. A simple computation gives

$$J_s = \omega^2 \frac{6 - \sqrt{26}}{5} \quad (9)$$

Introducing a normalized time and action

$$y = \frac{J}{J_s}, \quad \tau = \epsilon^2 \frac{J_s}{4} t \quad (10)$$

the Fokker-Planck equation reads

$$\frac{\partial \rho_0}{\partial \tau} = \frac{\partial}{\partial y} y^3 \left[1 + \frac{21}{8} \frac{J_s}{\omega^2} y \right] \frac{\partial \rho_0}{\partial y} \quad (11)$$

In figures 1 we compare the distribution function of the energy, computed by numerical simulation with the solution of the Fokker-Planck equation at $\tau = .01$. The initial condition was a narrow gaussian in the energy centered at the middle of the dynamic aperture, the number of particles in the numerical simulation was 40,000. In figure 2 we show the same distribution function as in fig. 1, but after a longer time ($\tau = .05$). The first order perturbative calculation of the diffusion coefficient gives an agreement of $\sim 20\%$ while the second order gives almost a best fit (remark that we had no adjustable parameters).

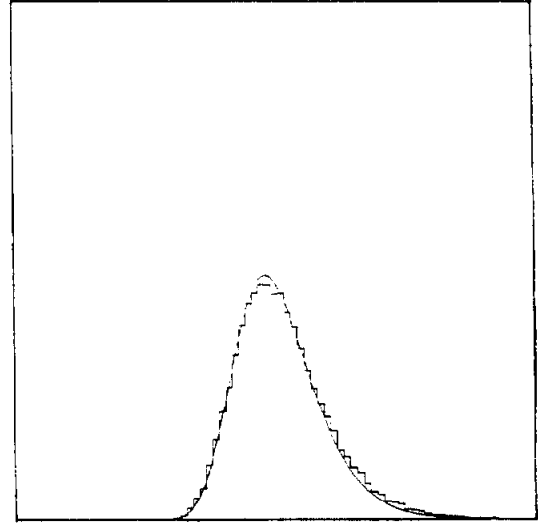


Figure 1: The distribution function for the energy when the linear frequency is stochastically perturbed: comparison between the numerical simulation and the solution of the Fokker-Planck's equation (smooth curve) at $\tau = .01$

We have considered the same hamiltonian system with a slow periodic modulation $\xi(t) = \cos \Omega t$ with $\Omega \ll \omega$. According to the adiabatic theory the separatrix is slowly pulsating with the same frequency Ω . We have distributed the particles uniformly half way on the ring swept by the separatrix along an unperturbed trajectory. The population has a low spread due to the modulation; when the separatrix crosses a particle then the particle is kicked off towards infinity. According to this picture, which becomes exact in the limit $\Omega \rightarrow 0$, the population remains constant until the encounter with the separatrix occurs after a quarter of period $T = \pi/(2\Omega)$, and afterwards it vanishes in a very short time interval. When Ω is rather small $\Omega/\omega = 5 \times 10^{-4}$ this phenomenon is observed with a good accuracy (Fig. 3). When Ω/ω is increased to 10^{-1} the adiabatic theory is no longer applicable and after a first rapid decrease, a long queue is observed (Fig. 4) due to the chaotic region generated by the separatrix.

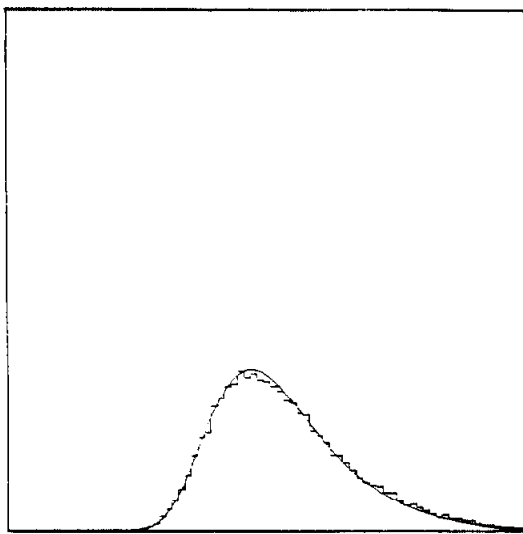


Figure 2: The distribution function for the energy when the linear frequency is stochastically perturbed: comparison between the numerical simulation and the solution of the Fokker-Planck's equation (smooth curve) at $\tau = .05$

III. CONCLUSIONS

Certainly the proposed model is a very crude description of the ripple induced diffusion in particle accelerator however this can be certainly useful to understand the effect of a stochastic or periodic perturbation of the linear frequency of a non-linear Hamiltonian systems. The main defect is the absence of resonances, which are taken into account when a discrete description of the lattice is given by an area preserving map. An extension of the previous results to a discrete model is perhaps possible but mathematically much more difficult to be justified. Certainly simulations can be carried out and the present analysis will be helpful in interpreting them.

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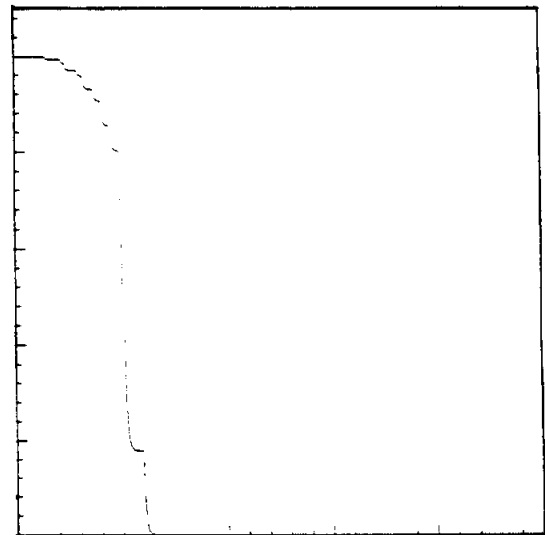


Figure 3: Number of particles versus time: the initial population is 10.000, the ratio Ω/ω is .0005 and each unity on the x-axis corresponds to 5000 time unities

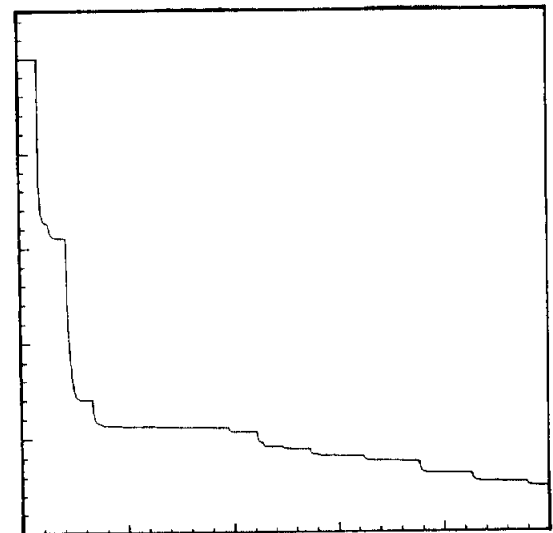


Figure 4: Number of particles versus time: the initial population is 10.000, the ratio Ω/ω is .1 and each unity on the x-axis corresponds to 5000 time unities